ORTHOGONAL HARMONIC FUNCTIONS IN SPACE[1]

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1. **Introduction.** The origin of this paper was an attempt to generalize to three dimensions certain simple properties of harmonic functions in two dimensions. Let $u(x, y)$ and $v(x, y)$ be harmonic functions in two dimensions. We shall say that u and v are *orthogonal* in some region if the following condition is satisfied there,

$$u_x v_x + u_y v_y = 0. \tag{1}$$

Note that we designate partial derivatives by writing as subscripts the variables with respect to which the differentiations are made. It is easy to show that (1) is the necessary and sufficient condition that the product uv be harmonic. That such orthogonal pairs of harmonic functions exist in two dimensions, aside from the trivial pairs in which one of the functions is a constant, follows from the Cauchy-Riemann equations for the conjugate functions u and v,

$$u_x = v_y, \qquad u_y = -v_x,$$

from which (1) follows immediately. A further property in which we were interested is that in two dimensions for any given nonconstant harmonic function u, another nonconstant harmonic function v can be found orthogonal to it. Actually all such functions differ only by a constant factor and an additive constant; again the proof follows easily by using conjugate functions.

Now generalize to three dimensions. Let ∇^2 represent the usual operator $\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$. If two harmonic functions $u(x, y, z)$, $v(x, y, z)$ satisfy the relation

$$\nabla u \cdot \nabla v \equiv u_x v_x + u_y v_y + u_z v_z = 0, \tag{2}$$

in some region, we shall say that they are *orthogonal* in that region. Since

$$\nabla^2 uv = u\nabla^2 v + v\nabla^2 u + 2\nabla u \cdot \nabla v, \tag{3}$$

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we see that again the necessary and sufficient condition that the product of the harmonic functions u and v be harmonic in a region is that u and v be orthogonal in that region.

It should be noted that this definition of orthogonal functions in three dimensions is in effect a generalization of the notion of an analytic function of a complex variable (considered as a pair of functions u, v) to three dimensions. Many such generalizations exist, depending on just what two-dimensional property of analytic functions is used [**7**].

We might suspect from analogy with the two-dimensional case that given a harmonic function u in three dimensions we could always find a nonconstant harmonic function v orthogonal to it. In fact, it is shown in §2 that this is the case if the given function u is a function of only one or two variables. Moreover, this is also the case if u is a harmonic polynomial of the second degree, as is shown in §3. However, any possibility that this is still the case for the general harmonic polynomial of degree higher than two, and a fortiori for harmonic functions in general, is destroyed by the counterexample exhibited in §4. In that section some brief remarks are also made as to constructive solutions (at least from the viewpoint of theory) to the following two problems:

(1) determining whether or not for a given harmonic function $u(x, y, z)$ there exists a nonconstant orthogonal harmonic function, and

(2) determining all such functions in case one does exist.

There appears to have been no previous specific statement or recognition of the existence problems of orthogonal harmonic functions in three dimensions. However, the problem of the binary potential, as developed by Levi-Civita [**6**], is a related problem. This problem is to find all three-dimensional harmonic functions which can be expressed in the form $\phi(u, v)$, where $u(x, y, z)$ and $v(x, y, z)$ are independent. Even if the solution of this problem were available in explicit form, it is believed that the supplementary considerations needed to bring it to bear on the present problem would be more involved than the direct treatment here presented. Again our problem is also related to work of Delens [**2**], who, not knowing at the time of Levi-Civita's work, had studied the problem of the binary potential by another method.

2. **Harmonic functions of one or two variables.** Let us consider the problem of finding all the harmonic functions which are orthogonal to a given harmonic function of one variable, say $u(x)$. Since u is harmonic we have $u_{xx}=0$, so that $u=a_1x+a_2$, where the a's are

arbitrary constants. Let $v(x, y, z)$ be harmonic. If we require v to be orthogonal to u, we have $\nabla u \cdot \nabla v = 0 = a_1 v_x$. Therefore v is a function of y and z only. Thus the class of harmonic functions orthogonal to a nonconstant harmonic function of one variable is the class of all harmonic functions in the other two variables; further, the given function of one variable must be a linear function of that variable.

Next we consider the problem of finding all the harmonic functions which are orthogonal to a given harmonic function of two variables, say $u(x, y)$. From Equation (2) we have

$$v_x u_x + v_y u_y = 0. \tag{4}$$

The relevant set of ordinary differential equations for solving this partial differential equation in v is [**5**, pp. 74–76]

$$dx/u_x = dy/u_y = dz/0. \tag{5}$$

A first integral is $z = c_1$. We then use the first two members of set (5) but introduce a function $w(x, y)$ conjugate to $u(x, y)$. Then $u_x = w_y$ and $u_y = -w_x$, so that we have $dx/w_y = -dy/w_x$, which reduces to $w_x dx + w_y dy = 0$, or $w(x, y) = c_2$ is another first integral of set (5). Therefore the general integral of Equation (4) is $v = F(w, z)$. If v is to be harmonic we must have $\nabla^2 v = 0$ which is

$$F_{zz} + (w_x^2 + w_y^2) F_{ww} = 0. \tag{6}$$

Let us choose independent variables x, z, w. They are really independent since the Jacobian $D(x, z, w)/D(x, y, z) = -w_y$. After the change of variables has been made, differentiate equation (6) with respect to x. The result is $(w_x w_{xx} + w_y w_{xy}) F_{ww} = 0$. If $\partial(w_x^2 + w_y^2)/\partial x = 0$, we would choose y, z, w as independent variables. We conclude then that $F_{ww} = 0$, except when $w_x^2 + w_y^2$ is a constant.

We first consider the case in which $w_x^2 + w_y^2 = u_x^2 + u_y^2$ is not a constant. Then $F_{ww} = 0$ and from equation (6) $F_{zz} = 0$. From these two conditions we have $F = c_1 zw + c_2 z + c_3 w + c_4$, where the c's are arbitrary constants.

Thus for a given harmonic function $u(x, y)$ actually containing both x and y (that is, neither u_x nor u_y is identically equal to zero) and for which $u_x^2 + u_y^2$ is not a constant, all the harmonic functions orthogonal to it are given by the formula

$$F = c_1 zw + c_2 z + c_3 w + c_4, \tag{7}$$

where $w(x, y)$ is a function conjugate to $u(x, y)$ and the c's are arbitrary constants.

We now consider the case in which for harmonic $u(x, y)$ in some region R

$$u_x^2 + u_y^2 = k^2, \tag{8}$$

where k is a constant. We shall show that then in R

$$u = k(x \cos \beta - y \sin \beta) + c, \tag{9}$$

where β and c are arbitrary constants. Let the analytic function $f=u+is$, where s is a function conjugate to u. Then $t=df/dz=u_x+is_x$ is also analytic and $|t|^2=u_x^2+s_x^2=u_x^2+u_y^2=k^2$, by the Cauchy-Riemann equations. Since the modulus of s is constant, so is s itself [**8**, §3.51, p. 120], whence u_x and u_y are constants, so that u has the form given in formula (9).

The formula (9) is essentially a one variable case. For we can make the substitution

$$x' = x \cos \beta - y \sin \beta,$$
$$y' = x \sin \beta + y \cos \beta,$$

so that u becomes $kx'+c$ and is still harmonic in the x', y', z system. If $k \neq 0$, the class of harmonic functions orthogonal to u was shown to be the class of all harmonic functions in y' and z, represented by $r(y', z)=r(x \sin \beta+y \cos \beta, z)$. We have thus shown that a nonconstant harmonic function $u(x, y)$ for which $u_x^2+u_y^2=k^2$ must be of the form (9) and that the class of harmonic functions orthogonal to $u(x, y)$ is the class of all functions of the form $r(x \sin \beta+y \cos \beta, z)$ where $r(u, z)$ is a harmonic function of u and z.

3. **Second degree harmonic polynomials.** We shall now show that a nonconstant orthogonal harmonic function can be found for each harmonic polynomial of the second degree in the three variables x, y, z. Since the properties of orthogonality and of being harmonic are invariant under translations and rotations, we simplify the problem by making use of the well-known result familiar from analytic geometry that the general second degree polynomial in x, y, z can be transformed by rotations and translations into one of the two forms

$$ax^2 + by^2 + cz^2 + d, \tag{10}$$

$$mx^2 + ny^2 + pz. \tag{11}$$

We therefore need to consider only harmonic polynomials of these two types, under the assumptions that $a+b+c=0$, or $m+n=0$. Since it is easily seen that the harmonic polynomials xyz for the form (10) and xy for the form (11) are then orthogonal to (10) and (11) respectively, we have thus shown that a nonconstant orthogonal harmonic function can be found for each three-dimensional second degree harmonic polynomial. Lest one be tempted to think that an orthogonal harmonic function in this case need necessarily be a polynomial, the fact that $\tan^{-1}(x/y)$ is an orthogonal harmonic function for the special case of (10), $x^2+y^2-2z^2$, will show otherwise.

4. **The counterexample.** In §1 we stated that we would prove by producing a counterexample that it is not in general true that given a nonconstant three-dimensional harmonic function u, in particular a third degree harmonic polynomial, we could always find a nonconstant harmonic function v orthogonal to it. The counterexample, it will turn out, is the harmonic polynomial

$$u = x^3 + y^3 - 3xz^2 - 3yz^2.$$

First let us find all the solutions, v, of Equation (2) which becomes

$$(x^2 - z^2)v_x + (y^2 - z^2)v_y - 2z(x + y)v_z = 0. \tag{12}$$

It follows from a standard theorem [5, pp. 74–76] that if we find two independent analytic first integrals f_1 and f_2 which are valid in some region D, of the following set of ordinary differential equations,

$$\frac{dx}{x^2 - z^2} = \frac{dy}{y^2 - z^2} = \frac{dz}{-2z(x + y)}, \tag{13}$$

then all the solutions of (12) in this same region D are of the form $v=F(f_1, f_2)$, where F has continuous partial derivatives of the first order.

We consequently shall study the set (13). An integrable combination is

$$\frac{dx - dy}{x - y} = \frac{dz}{-2z},$$

whence

$$(x - y)^2 z = c_1 \tag{14}$$

is a first integral. If we solve it for z, we find $z=c_1/(x-y)^2$. If we substitute this value in the first two equations of (13), we find

$$\frac{dx}{x^2 - c_1^2/(x - y)^4} = \frac{dy}{y^2 - c_1^2/(x - y)^4}$$

or

$$[y^2 - c_1^2/(x - y)^4]dx - [x^2 - c_1^2/(x - y)^4]dy = 0.$$

An integrating factor is $1/(x-y)^2$ so that

$$\left[\frac{y^2}{(x-y)^2}-\frac{c_1^2}{(x-y)^6}\right]dx-\left[\frac{x_2^2}{(x-y)^2}-\frac{c_1^2}{(x-y)^6}\right]dy=0$$

is an exact differential and the resulting integral is

$$-y-\frac{y^2}{x-y}+\frac{c_1^2}{5(x-y)^5}=k_1. \tag{15}$$

If in (15) we substitute the value of c_1 from (14), we find

$$\frac{z^2-5xy}{x-y}=c_2.$$

Thus $v=F[(x-y)^2z,\ (z^2-5xy)/(x-y)]$ is the general integral of Equation (12).

It will be more convenient in what follows to use the following equivalent general integral

$$\begin{aligned} v &= F[(x-y)^2z,\ z(z^2-5xy)^2] \\ &= F(s,t), \end{aligned} \tag{16}$$

where

$$s=(x-y)^2z, \qquad t=z\,(z^2-5xy)^2.$$

Note that F is an arbitrary function, continuous and with continuous first derivatives with respect to s and t. Since we are to test v for harmonicity, it is intuitively reasonable also to require that F have continous second derivatives with respect to s and t. However, we shall now make a digression to consider in rigorous detail what requirements F must satisfy when we require v to be a harmonic function.

Suppose we have $v=F(s,t)$, where s and t are independent and each is a continuous function of x, y and z. Must F be a continuous function of s and t for v to be a continuous function of x, y and z? This is almost obvious for if F were not continuous, then it would not change only by small increments for small changes in s and t caused by small changes in x, y and z, and v would not be continuous.

There is also the question as to the differentiability of F. Let $v(x,y,z)=F(s,t)$ where v is continuous together with its derivatives with respect to x, y and z through the second order. Also F is a continuous function of s and t, and s and t are independent and each is a continuous function of x, y and z, with continuous partial derivatives through the second order with respect to these variables. Does it follow that F has continuous derivatives with respect to s and t

through the second order? This may seem reasonably certain on an intuitive basis and we can give a formal proof as follows.

Let

$$s = s(x, y, z),$$
$$t = t(x, y, z).$$

Since s and t are independent, an inversion of these equations will exist, at least in some finite region, by the theory of implicit functions [**4**, §§25–26, pp. 45–51]. That is, we may write

$$x = f_1(s, t, z), \qquad y = f_2(s, t, z),$$

where f_1 and f_2 are continuous and possess continuous partial derivatives through the second order. Specifically, in the case we are considering, the Jacobian $D(s, t)/D(x, y) = -20z^2(x^2-y^2)(z^2-5xy)$, so that the inversion is possible except on the surfaces $z=0$, $x^2-y^2=0$, $z^2-5xy=0$.

Then $v(x, y, z) = v(f_1, f_2, z) = v^*(s, t, z)$, where v^* is what $v(x, y, z)$ becomes when f_1 and f_2 are substituted for x and y. It will turn out that $v^*(s, t, z)$ does not depend on z.

Now we shall show that the derivatives of v^* with respect to s and t exist through the second order and are continuous. Since v, f_1, and f_2 each possess continuous first order derivatives with respect to their arguments, the chain rule for compound functions [**1**, pp. 69–73] holds and we have

$$v_s^* = v_x \frac{\partial f_1}{\partial s} + v_y \frac{\partial f_2}{\partial s},$$

$$v_t^* = v_x \frac{\partial f_1}{\partial t} + v_y \frac{\partial f_2}{\partial t}.$$

Therefore the first derivatives of $v^*(s, t, z)$ with respect to s and t exist.

Further, since the quantities v_x, v_y, $\partial f_1/\partial s$, $\partial f_2/\partial s$, $\partial f_1/\partial t$, $\partial f_2/\partial t$, which occur in the formulas for v_s^* and v_t^*, have been assumed to possess continuous partial derivatives with respect to their arguments, the second derivatives of v^* with respect to s and t obviously exist. Thus both the first and second derivatives of v^* with respect to s and t exist and are continuous.

Now $F(s, t) = F^*(x, y, z) = v(x, y, z) = v^*(s, t, z)$, where F^* is what $F(s, t)$ becomes when $s(x, y, z)$ and $t(x, y, z)$ are substituted in it. Therefore $F(s, t) = v^*(s, t, z)$. Thus $v^*(s, t, z)$ does not contain z explicitly and may be written $v^*(s, t)$. Then $v^*(s, t)$ and $F(s, t)$ are the same function. But since $v^*(s, t)$ has been shown to have continuous partial derivatives through the second order with respect to s and t, $F(s, t)$ likewise does, which is what we wished to prove.

We are interested only in those general integrals (16) which are harmonic. We therefore calculate $\nabla^2 v$ from (16) and find

$$\begin{aligned}\nabla^2 v = {} & 4zF_s + 10z(5y^2 + 5x^2 + 2z^2 - 6xy)F_t \\ & + [8z^2(x - y)^2 + (x - y)^4]F_{ss} \\ & + [10(x - y)^2(z^2 - 5xy)(5z^2 - xy)]F_{st} \\ & + 25(z^2 - 5xy)^2[4z^2(x^2 + y^2) + (z^2 - xy)^2]F_{tt}.\end{aligned} \tag{17}$$

Next we replace the independent variables x, y, z by z, s, t. That the latter are independent variables can be checked by computing the Jacobian $D(z, s, t)/D(x, y, z)$. It equals $-20z^2(x^2-y^2)(z^2-5xy)$, which is not identically equal to zero. With the new independent variables, the equation $\nabla^2 v = 0$ becomes

$$\begin{aligned}4z^3F_s + (50sz^2 + 28z^5 \mp 8t^{1/2}z^{5/2})F_t & \\ + (8sz^3 + s^2)F_{ss} + (2st \pm 48st^{1/2}z^{5/2})F_{st} & \\ + (100stz^2 + 56tz^5 + t^2 \mp 32t^{3/2}z^{5/2})F_{tt} & = 0,\end{aligned} \tag{18}$$

where the plus and minus signs are to be matched as indicated. Let us rewrite equation (18) as a power series in $z^{1/2}$. We find, by equating the coefficients of the powers of $z^{1/2}$ to zero, that

$$\begin{aligned} s^2F_{ss} + 2stF_{st} + t^2F_{tt} &= 0, \\ sF_t + 2stF_{tt} &= 0, \\ t^{1/2}F_t - 6st^{1/2}F_{st} + 4t^{3/2}F_{tt} &= 0, \\ F_s + 2sF_{ss} &= 0, \\ F_t = 2tF_{tt} &= 0. \end{aligned} \tag{19}$$

From the second or the last equation we have

$$F = 2t^{1/2}\phi_1(s) + \phi_2(s),$$

where ϕ_1 and ϕ_2 are functions only of s. From the fourth equation we have

$$F = 2s^{1/2}\phi_3(t) + \phi_4(t),$$

where ϕ_3 and ϕ_4 are functions only of t. We therefore have

$$F = k_1s^{1/2}t^{1/2} + k_2s^{1/2} + k_3t^{1/2} + k_4, \tag{20}$$

where the k's are arbitrary constants. Differentiate this value of F

and substitute in the first and third equations of (19). These become respectively

$$-k_2 s^{1/2}/4 - k_3 t^{1/2}/4 = 0, \qquad -2k_1 s^{1/2} - k_3/2 = 0.$$

Since s and t are independent variables, we have $k_1 = k_2 = k_3 = 0$, whence from equation (20), $F = k_4$, a constant.

It thus follows that v is not harmonic unless F is a constant. Therefore no nonconstant harmonic function v exists which satisfies $\nabla^2 uv = 0$, where u is the harmonic function $x^3 + y^3 - 3xz^2 - 3yz^2$.

It is true that we have had to exclude more or less implicitly certain surfaces to avoid having zero denominators or to avoid a zero value for the Jacobian of the transformation used. The surfaces thus excluded were $z = 0$, $x^2 - z^2 = 0$, $y^2 - z^2 = 0$, $x + y = 0$, $x - y = 0$, $s = 0$ (that is, $(x-y)^2 z = 0$), $t = 0$ (that is, $z(z^2 - 5xy)^2 = 0$). Since these equations do not define a three-dimensional region, we have not lost any harmonic function by excluding them, since, by its very definition, a harmonic function must be given in at least a (three-dimensional) neighborhood.

It may be useful to point out here that the method used in checking the counterexample can be generalized to provide a method of determining whether or not for a given harmonic function $u(x, y, z)$ there exists a nonconstant orthogonal harmonic function and of finding all such functions in case one does exist. One starts with the partial differential equation (2) and finds two independent first integrals s, t, of the related set of ordinary differential equations

$$dx/u_x = dy/u_y = dz/u_z,$$

so that $v = F(s, t)$ is the general integral of (2). Then form the equation $\nabla^2 v = 0$, which is a linear equation in the five first and second derivatives of F. Next make the appropriate change of variables, say from (x, y, z) to (s, t, z) and from the expansion of the equation in powers of z obtain, by equating to zero the coefficient of each power of z, a set of linear equations in the five derivatives of F with coefficients which are functions of s and t. This set of equations determines the relevant class of orthogonal harmonic functions. A similar method was used by Delens [2, pp. 78–82]. An alternative procedure is to use the four equations found by differentiating repeatedly with respect to z the new form of the equation $\nabla^2 v = 0$, as well as this new equation itself. If the determinant of these five equations, which are linear in the five derivatives of F, is not zero, no nonconstant orthogonal harmonic function exists. Otherwise we can again find an analogous set of linear equations in the five derivatives of F. These methods, while theoretically constructive, may at times be unworkable practically. Some further details are given in the author's thesis [**3**, §7 of Chap. II, pp. 29–32].

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