

A STURMIAN THEOREM FOR PARTIAL DIFFERENTIAL EQUATIONS

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1. **Introduction.** In a paper with which the reader is presumed to be familiar [1], Hartman and Wintner have shown how the classical theorems of Sturm concerning the zeros of solutions may be extended to a wide class of partial differential equations. The result to be established here is related to that of [1], though neither includes the other. Whereas [1] compares the equations

$$L(u) + \phi u = 0, \quad L^*(u) + \phi^* u = 0$$

with different operators L , L^* and functions ϕ , ϕ^* , the present analysis requires $L = L^*$. By way of compensation, however, it will be seen that the assumptions on L and on the region are somewhat weaker than those of [1]. For this reason the methods of Hartman and Wintner are not directly applicable; and in fact, our procedure has little in common with theirs.

2. **Definitions and notations.** Let a nonlinear operator L be defined by

$$(1) \quad L(w) = \sum a_{ij}(x_k, w_k/w) w_{ij}$$

where

$$\begin{aligned} w &= w(x_1, x_2, \dots, x_n), \\ w_i &= \partial w / \partial x_i, \\ w_{ij} &= \partial^2 w / \partial x_i \partial x_j \end{aligned}$$

and where $a_{ij}(x_k, w_k/w)$ is an abbreviation for

$$a_{ij}(x_1, x_2, \dots, x_n; w_1/w, w_2/w, \dots, w_n/w).$$

It is supposed that a_{ij} is a well-defined function of its $2n$ arguments for all values relevant to the ensuing discussion. Similar remarks apply to the functions ϕ and ϕ^* introduced in the sequel; e.g.,

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$$\phi(x_k, w_k/w) = \phi(x_1, x_2, \dots, x_n; w_1/w, w_2/w, \dots, w_n/w).$$

To avoid difficulty when $w=0$, and to postulate the required smoothness, we introduce the following definition:

DEFINITION 1. In a region R , the function w is an *admissible partial* solution of

$$(2) \quad L(w)/w \geq \phi(x_k, w_k/w), \quad (x_1, \dots, x_n) \in R$$

if w_i is differentiable in R for each i , and if (2) holds at every point of R where $w \neq 0$.

The differentiability ensures $w_{ij} = w_{ji}$ by a known theorem of Young [3], and this in turn yields the usual condition

$$(3) \quad (w_{ij}) \leq 0$$

at a maximum of w . (Here the notation (3) means that the matrix (w_{ij}) is negative semidefinite; and similarly in other cases.)

It is desirable to specify the behavior of w on the boundary of R without, however, requiring continuous boundary values.

DEFINITION 2. If w is defined in a region R , we say

$$\inf \sup w \leq m \text{ at the boundary,}$$

when there is a sequence of compact subsets

$$R_n \subset R_{n+1} \subset R, \quad \lim R_n = R$$

such that

$$\sup w \leq m + 1/n,$$

the sup being taken over the boundary of R_n .

It is not required that $w \leq m$ in a neighborhood of the boundary; indeed, $\lim \sup w = \infty$ at the boundary is permissible. The notation "inf sup" suggests that one takes the sup on the boundary of R_n , but the inf with respect to the various choices of R_n that are possible.

3. **The theorem.** These preliminaries enable us to state the following Sturmian theorem, analogous to that of [1]:

THEOREM. *Let u be an admissible partial solution of the boundary-value problem*

$$(4) \quad \begin{aligned} L(u)/u &\geq \phi(x_k, u_k/u), \\ \inf \sup |u| &\leq 0 \text{ at the boundary} \\ u &\neq 0 \end{aligned}$$

in a region R , where L is the operator (1). Suppose $(a_{ij}) \geq 0$, and let ϕ^ be a function such that*

$$\phi^*(x_k, s_k) < \phi(x_k, s_k)$$

for all relevant values of the $2n$ arguments. Then every admissible partial solution of

$$(5) \quad L(v)/v \leq \phi^*(x_k, v_k/v)$$

satisfies $\inf |v| = 0$ in R .

If $\inf |v| > 0$, we may define w by

$$u = vw.$$

Since $u \neq 0$, and since $\inf \sup |u| \leq 0$ at the boundary, it is easily established that w attains a positive maximum or negative minimum at an interior point of R . At this point we have $uv \neq 0$, so that (4) and (5) hold. Also

$$\begin{aligned} u_i/u &= v_i/v, \\ u_{ij}/u &= v_{ij}/v + w_{ij}/w \end{aligned}$$

and hence, at the point in question,

$$\begin{aligned} \phi(x_i, v_i/v) &= \phi(x_i, u_i/u) \leq L(u)/u \\ &= \sum a_{ij}(x_k, u_k/u)u_{ij}/u \\ &= \sum a_{ij}(x_k, v_k/v)[v_{ij}/v + w_{ij}/w] \\ &= L(v)/v + \sum a_{ij}w_{ij}/w \\ &\leq \phi^*(x_i, v_i/v) + \sum a_{ij}w_{ij}/w. \end{aligned}$$

Since $(a_{ij}) \geq 0$ everywhere, and since

$$(w_{ij}/w) \leq 0$$

at the extremum, the theorem of Schur and Fejér [2] implies

$$\sum a_{ij}w_{ij}/w \leq 0.$$

Thus, the condition $\phi > \phi^*$ is violated; and this completes the proof.

4. Concerning nonlinearity. In many problems of the sort considered here the nonlinear character is really fictitious; for, an equation such as (2) becomes linear when one substitutes in the values of w and w_i . If u and v satisfy nonlinear equations of the type contemplated in the theorem then they satisfy linear homogeneous equations.

It is not possible to proceed thus in the proof, for the following reason. When we carry out that process for u and v we get two equations which are both linear, to be sure; but the equations are otherwise quite unrelated. The coefficients for the v equation are evaluated

by substituting v , while those of the u equation are found from u . Since there is no reason to expect $u=v$, part of the hypothesis disappears in the process, and the proof cannot be completed. For the same reason, the hypothesis does not ensure $L(v)/v \leq L(u)/u$.

Actually, the theorem is false when the functional form of the coefficients is unrestricted. Taking $n=2$, $x_1=x$, $x_2=y$, let an operator L be defined by

$$L(u) = u_{xx} \left[1 + 4 \left(\frac{u - \cos 2x \cos 2y}{\cos x \cos y + \cos 2x \cos 2y} \right)^2 \right]$$

in analogy with (1) and let R be the square

$$|x| < \pi/4, \quad |y| < \pi/4.$$

Then a_{ij} is well-defined, $a_{11} \geq 1$, $a_{12} = a_{21} = a_{22} = 0$, and hence $(a_{ij}) \geq 0$. The function

$$u = \cos 2x \cos 2y$$

satisfies

$$L(u)/u = -4, \quad u = 0 \text{ at the boundary.}$$

The function

$$-v = \cos x \cos y$$

satisfies

$$L(v)/v = -5, \quad v \neq 0 \text{ in } \bar{R}.$$

Hence the theorem is not valid for this operator L , and the special form of the coefficients must actually play a role in the analysis.

5. Concluding remarks. We wish to indicate how the condition $\phi^* < \phi$ may be replaced by $\phi^* \leq \phi$, though detailed proofs are not presented here. The operator L is said to be *weakly definite* if $(a_{ij}) \geq 0$ and if, for each compact subset S of R , there is a constant vector c_i such that

$$\sum a_{ij} c_i c_j > 0 \text{ throughout } S.$$

When L is weakly definite in this sense, and when

$$a_{ij} \in C^{(1)}, \quad \phi \in C^{(1)}, \quad \phi^* \in C^{(1)}, \quad u \in C^{(2)}$$

it is not difficult to show that $\phi \leq \phi^*$ is sufficient to yield the conclusion of the theorem. In fact, the inequality

$$vL(u) - uL(v) \geq \phi(x_k, u_k/u, v_k/v)$$

with mild restrictions on ϕ ensures that u/v satisfies the maximum principle; and this is the property which underlies the present analysis.

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A NOTE ON LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Let

$$(1) \quad \frac{dx}{dt} = A(t)x,$$

where x is an n -column vector

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

and $A = (a_{ij}(t))$ where $a_{ij}(t)$ are continuous real valued functions of time ($-\infty < t < +\infty$). Let $y^1(t), \dots, y^n(t)$ be any n -linearly independent solutions of (1) defined for all t . Let $B^1(t), \dots, B^n(t)$ be the n normal-orthogonal vectors obtained from the set $\{y^i\}$ by the Gram-Schmidt orthogonalization process. Let $B(t)$ be the orthogonal matrix whose j th column is $B^j(t)$, and introduce a new variable u (an n -column vector) defined by

$$(2) \quad x = B(t)u.$$

u satisfies the linear differential equation

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