MATRIX SUMMABILITY IN F-FIELDS

J. B. ROBERTS

In this paper we consider some aspects of the theory of summability in fields which are complete with respect to a non-Archimedean valuation. We call such fields *F-fields*. In particular we give necessary and sufficient conditions in order that certain linear summability methods preserve convergence and limits.

1. **Definitions.** The infinite matrix $A = (a_{ij})$ will be associated with the

$$\left\{\begin{matrix}\text{sequence to sequence}\\ \text{series to sequence}\\ \text{series to series}\end{matrix}\right\} \text{transformation} \left\{\begin{matrix}[u_n] \to [u_n']\\ \sum_{i=0}^{\infty} u_i \to [u_n']\\ \sum_{i=0}^{\infty} u_i \to \sum_{i=0}^{\infty} u_i'\end{matrix}\right\}$$

when $u_n' = \sum_{i=0}^{\infty} a_{in} u_i$ for all $n \geqq 0$.

If the sequence to sequence transformation associated with A is such that $[u_n']$ exists and converges whenever $[u_n]$ converges we call A a *K matrix*. If $[u_n']$ exists and converges to $\lim_{n\to\infty} u_n$ whenever this limit exists we call A a *T matrix*. Similar names for the series to sequence and the series to series matrices are K_1, T_1 and K_2, T_2 *matrices* respectively. (Our K_1, T_1, T_2 matrices are sometimes referred to as β, γ, α matrices respectively. See [3].)

2. **The main theorem.** For ease in stating the main theorem we set forth three propositions. We use $|a|$ for the valuation of a in the F-field under consideration.

(i) $|a_{ij}| < H$ for $i, j \geqq 0$ and some real H;

(ii) $\sum_{i=0}^{\infty} a_{ij}$ exists for all j and tends to γ as $j \to \infty$;

(iii) for $i \geqq 0$, $\lim_{j\to\infty} a_{ij}$ exists and equals α_i.

THEOREM. *The matrix $A = (a_{ij})$ is a:*

(1) *K matrix if and only if* (i), (ii), (iii);

(2) *T matrix if and only if* (i), (ii), (iii) *and $\gamma = 1$ and all $\alpha_i = 0$;*

(3) *K_1 matrix if and only if* (i), (iii);

(4) *T_1 matrix if and only if* (i), (iii) *and all $\alpha_i = 1$;*

(5) *K_2 matrix if and only if* (i) *and $\sum_{j=0}^{\infty} a_{ij}$ exists for $i \geqq 0$;*

(6) *T_2 matrix if and only if* (i) *and $\sum_{j=0}^{\infty} a_{ij} = 1$ for $i \geqq 0$.*

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Furthermore if A is a:

(a) *K matrix then $u_n' \to \gamma u + \sum_{i=0}^{\infty} \alpha_i(u_i - u)$;*

(b) *K_1 matrix then $u_n' \to \alpha_0 \sum_{i=0}^{\infty} u_i + \sum_{i=0}^{\infty} ((\alpha_{i+1} - \alpha_i) \sum_{j=i+1}^{\infty} u_j)$.*

Except for the proofs of the necessity of (i) in each of (1)–(6) the proofs are not difficult and are quite similar to the proofs in the complex number field. A proof of (2) was recently given for p-adic number fields. (See [1].) In §4 we will prove a lemma which at once furnishes us with a proof of the necessity of (i) in all of the six parts of the theorem. For complete proofs of all parts of the theorem see [5]. In the next section we introduce some concepts used in the proof of the lemma mentioned above.

3. F-**Banach spaces.** A vector space S over the F-field F, with valuation $|\cdot|$, is an *F-Banach space* if there exists a real valued function $\|\cdot\|$ satisfying

(i) $\|x\|$ exists and is $\geqq 0$ for all $x \in S$ and $\|x\| = 0$ only for $x = 0$;

(ii) $\|x+y\| \leqq \|x\| + \|y\|$;

(iii) $\|cx\| = |c| \cdot \|x\|$ for all $c \in F$ and $x \in S$;

(iv) S is complete with respect to $\|\cdot\|$.

Following Banach [2] we denote the collections of bounded sequences and null sequences by (m) and (c_0) respectively. Defining $\|x\|$ for $x = [x_n] \in (m)$ to be $\sup_n |x_n|$ we see that both (m) and (c_0) are F-Banach spaces.

Also the following analogue of the Banach-Steinhaus theorem is valid.

Let $[U_\alpha]$ be a family of bounded linear operators defined on an F-Banach space $\tilde{M}$ to an F-Banach space $\tilde{N}$ and let M_{U_α} be the bound of U_α. Then if $\sup_\alpha \|U_\alpha(x)\|$ is finite for all $x \in \tilde{M}$ the set $[M_{U_\alpha}]$ is bounded.

For proofs and further discussion of these facts see [**5**, Chapter 5].

4. **The key lemma.**

LEMMA. *If for arbitrary $[x_n] \in (c_0)$, $x_n' = \sum_{i=0}^{\infty} a_{in} x_i$ exists for all $n \geqq 0$ and $[x_n'] \in (m)$ then there exists a real number H such that $|a_{ij}| < H$ for $i, j \geqq 0$.*

PROOF. (a) We first show that for each $j \geqq 0$, $\sup_i |a_{ij}|$ exists. Assume the contrary. Let $[x_n] \in (c_0)$ with no $x_n = 0$. Then there exists a strictly increasing sequence of positive integers $[i_k]$ such that $|a_{i_k j_0}| > |x_k|^{-1}$ for $k \geqq 0$. Define $y_{i_k} = x_k$, $y_i = 0$ otherwise. Then $[y_n] \in (c_0)$ and so $\sum_{k=0}^{\infty} a_{i_k j_0} x_k$ does not exist. This contradiction proves $\sup_i |a_{ij}|$ exists for $j \geqq 0$.

(b) Let $[x_n] \in (c_0)$. Then by hypothesis $\sum_{i=0}^{\infty} a_{in}x_i$ exists for $n \geqq 0$. Let $U_n(x) = \sum_{i=0}^{\infty} a_{in}x_i$. This U_n is a homogeneous linear operator on (c_0) to F. By (a), $\sup_i |a_{in}|$ exists and therefore $|U_n(x)| = |\sum_{i=0}^{\infty} a_{in}x_i| \leqq \sup_i |a_{in}x_i| \leqq M_n\|x\|$. Hence U_n is continuous (see [4]).

We show next that $M_n = M_{U_n}$, where M_{U_n} is the bound of U_n. Clearly $M_n \geqq M_{U_n}$. Suppose $M_n > M_{U_n}$. Let $M_{U_n} = M_n - d$. Choose i_0 such that $|a_{i_0 n}| > M_n - d$. Define the sequence $[y_n]$ so that $y_{i_0} = 1$, $y_i = 0$ otherwise. Then $|U_n(y)| = |a_{i_0 n}| > M_n - d = M_{U_n}$. But by definition of M_{U_n}, $|U_n(y)| \leqq M_{U_n}\|y\| = M_{U_n}$. This contradiction shows $M_n = M_{U_n}$.

By hypothesis if $x = [x_n]$ is in (c_0) then $[U_n(x)]$ is in (m). Hence $\sup_n |U_n(x)| < \infty$. Since the U_n are linear operators on (c_0) to F with $\sup_n |U_n(x)| < \infty$ for all $x \in (c_0)$ we conclude from the Banach-Steinhaus theorem that $[M_{U_n}] \in (m)$. Hence if $M_{U_n} < H$ for all $n \geqq 0$, $\sup_j \sup_i |a_{ij}| < \sup_j M_j = \sup_j M_{U_j} < H$ and the lemma is proved.

References

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Reed College