COMPLETE ORTHONORMAL SEQUENCES OF FUNCTIONS
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Given a region $G$ in the complex plane having a nonvoid interior and two complex valued functions, $f$ and $g$, defined in $G$, denote

$$(f, g)_G = \int \int_G f \overline{g} \, dx \, dy, \quad \|f\|_G = (f, f)_G^{1/2}.$$  

Let $D$ be a bounded plane domain and let $L^2(D)$ be the Hilbert space of all analytic functions in $D$ satisfying $\|f\|_D < \infty$. It can easily be shown that there exists a continuous, real valued function $m_D(z)$ in $D$ such that for any $f$ in $L^2(D)$ and any $z$ in $D$

$$(1) \quad \|f(z)\| \leq m_D(z)\|f\|_D,$$

(see, for example, [1, p. 5]).

Let $\phi_n$ be any complete orthonormal sequence in $L^2(D)$. Then from (1) follows

$$(2) \quad \sum_{n=1}^{\infty} \|\phi_n(z)\|^2 \leq [m_D(z)]^2,$$

and hence the convergence of the expansion for the Bergman reproducing kernel

$$(3) \quad K_D(z, w) = \sum_{n=1}^{\infty} \phi_n(z) \overline{[\phi_n(w)]},$$

where $[\phi_n(w)]$ indicates the complex conjugate of $[\phi_n(w)]$, (cf. [1, pp. 6 and 9]).

Let $K$ be a given compact subset of $D$. Then we might wish to try to approximate the kernel function in $K$ by using a finite series. Because of (2) and the fact that $m_D(z)$ is continuous, it is clear that given an $\epsilon > 0$ and a complete orthonormal sequence $\{\phi_n\}$, there exists an $N = N(D, K, \epsilon, \{\phi_n\})$ such that $m \geq N$ and $z, w \in K$ implies

$$\left| K_D(z, w) - \sum_{n=1}^{m} \phi_n(z) \overline{[\phi_n(w)]} \right| < \epsilon.$$  

It might be asked whether or not $N$ can be chosen independently of

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ORTHONORMAL SEQUENCES 493

\{\phi_n\}, i.e., if we can find an upper bound to the number of terms required in the series of (3) to approximate \(K_D(z, w)\) in \(K\), independently of which orthonormal sequence \{\phi_n\} is used.

In this note, we show that such an upper bound cannot exist. To do this, we first prove a result which the writer and several of those he has shown it to consider quite remarkable. If \(K\) is a compact subset of \(D\), and \(\varepsilon > 0\) is given, then there exists a complete orthonormal sequence \{\phi_n\} in \(L^2(D)\) such that for all \(z \in K\) and all \(n\), \(|\phi_n(z)| < \varepsilon\).

The method of proof used is of some interest since it is one of relatively few examples of the use of doubly orthogonal functions.

**Theorem 1.** Let \(D\) be a bounded domain in the complex plane, \(L^2(D)\) the Hilbert space of all analytic functions in \(D\) for which \(\|f\|_D < \infty\), and \(K\) a compact subset of \(D\). Let \(\varepsilon > 0\) be given. Then there exists a complete orthonormal sequence \{\phi_n(z)\} in \(L^2(D)\) such that \(\|\phi_n\|_K < \varepsilon\) for all \(n\).

**Proof.** We may assume that \(K\) is the closure of a domain contained in \(D\), for if not we merely enlarge \(K\) to \(K'\) satisfying this property.

Under these hypotheses we have the existence of a doubly orthogonal sequence of functions \{\psi_n(z)\} (cf. \[1, pp. 14–17\]), that is, a complete orthonormal sequence in \(L^2(D)\) satisfying

\[ (\psi_n, \psi_m)_K = \lambda_n \delta_{nm}, \quad \lambda_n \searrow 0. \]  

Indeed, \(\sum \lambda_n < \infty\), but we do not need this here. Since \(\lambda_n \to 0\), we have

\[ \frac{1}{m} \sum_{n=1}^{m} \lambda_n \to 0 \text{ as } m \to \infty. \]

Given \(\varepsilon > 0\), choose \(N\) such that

\[ \frac{1}{2^N} \sum_{n=1}^{2^N} \lambda_n < \varepsilon^2, \quad \lambda_n < \varepsilon^2 \text{ for } n > 2^N. \]

Let \(E_N = (\varepsilon_{ij})\) be a \(2^N \times 2^N\) matrix consisting entirely of elements \(\varepsilon_{ij} = \pm 1\) such that any two rows are orthogonal. The existence of such matrices is of course well known. One is easily constructed inductively, setting

\[ E_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad E_{n+1} = \begin{pmatrix} E_n & E_n \\ E_n & -E_n \end{pmatrix}. \]

Set

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\[ \phi_i = (1/2^N)^{1/2} \sum_{i=1}^{2^N} \epsilon_i \psi_i, \quad i = 1, 2, \ldots, 2^N. \]
\[ = \psi_i, \quad i > 2^N. \]

This set of functions is clearly orthonormal and complete in \( L^2(D) \).

From (4) and (5) we see that the desired conclusion, \( \|\phi_n\|_\infty < \epsilon \), holds for \( n > 2^N \), while if \( n \leq 2^N \), then
\[
\|\phi_n\|_\infty^2 = \frac{1}{2^N} \sum_{j=1}^{2^N} \lambda_j < \epsilon^2.
\]

**Corollary 1.** If \( K \) is a compact subset of the bounded plane domain \( D \) and if \( \epsilon > 0 \) is given, then there exists a complete orthonormal sequence \( \{\phi_n(z)\} \) in \( L^2(D) \) such that for all \( z \in K \) and all \( n \)
\[ \|\phi_n(z)\| < \epsilon. \]

**Proof.** Let \( G \) be a domain containing \( K \) and such that \( \overline{G} \subset D \). Let
\[
(6) \quad m = \max_{z \in K} m_G(z)
\]
where \( m_G(z) \) is defined as in (1).

From Theorem 1, construct an orthonormal sequence \( \{\phi_n\} \) in \( L^2(D) \) such that
\[ \|\phi_n\|_\sigma < \frac{\epsilon}{m} \]
for all \( n \). Then because of (1), for any \( z \in K \) and any \( n \)
\[ \|\phi_n(z)\| \leq m_G(z) \|\phi_n\|_\sigma < \epsilon. \]

**Corollary 2.** Let \( K \) be a compact subset of the bounded plane domain \( D \). Let \( 0 < \epsilon < \max K_D(z, z)/2 \) for \( z \in K \). Then there exists no integer \( N \) such that
\[ K_D(z, w) - \sum_{n=1}^{N} \phi_n(z) [\phi_n(w)]^{-} \leq K_D(z, z)/2 < \epsilon \]
for all \( z, w \in K \) and any complete orthonormal sequence \( \{\phi_n\} \) in \( L^2(D) \).

**Proof.** Let \( N \) be given. Choose \( z_0 \) in \( K \) such that \( K_D(z_0, z_0) > 2\epsilon \).
From Corollary 1, choose a complete orthonormal sequence \( \{\phi_n\} \) such that \( \|\phi_n(z_0)\|^2 < \epsilon/N \) for all \( n \). But then
\[ K_D(z_0, z_0) - \sum_{n=1}^{N} \phi_n(z_0) [\phi_n(z_0)]^{-} \geq K_D(z_0, z_0) - \sum_{n=1}^{N} \|\phi_n(z_0)\|^2 > \epsilon. \]
Finally we note that Theorem 1 is not confined to the space $L^2(D)$. For the proof we require only the existence of a doubly orthogonal sequence $\psi_n$ with $\lambda_n \to 0$. Thus, we have the result:

**Theorem 2.** Let $H_1$ and $H_2$ be two complete, separable Hilbert spaces and let $J: H_2 \to H_1$ be a linear mapping of $H_2$ into $H_1$. Suppose that $J$ is completely continuous. Then, given $\epsilon > 0$, there exists in $H_2$ a complete orthonormal sequence $\{\phi_n\}$ such that $\|J\phi_n\| < \epsilon$ for all $n$.

**Proof.** It is easily proved that if $J$ is completely continuous, there exists a doubly orthogonal sequence $\{\phi_n\}$, complete in $H_2$, and satisfying

\[
(\psi_n, \psi_m)_2 = \delta_{nm},
\]

\[
(J\psi_n, J\psi_m)_1 = \lambda_n \delta_{nm}, \quad \lambda_n \downarrow 0.
\]

The proof of this theorem then proceeds exactly as that of Theorem 1.

**Bibliography**


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