COMPLETE ORTHONORMAL SEQUENCES OF FUNCTIONS UNIFORMLY SMALL ON A SUBSET

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Given a region G in the complex plane having a nonvoid interior and two complex valued functions, f and g, defined in G, denote

$$(f, g)_G = \iint_G f\bar{g}\,dxdy, \qquad \|f\|_G = (f, f)_G^{1/2}.$$

Let D be a bounded plane domain and let $L^2(D)$ be the Hilbert space of all analytic functions in D satisfying $\|f\|_D < \infty$. It can easily be shown that there exists a continuous, real valued function $m_D(z)$ in D such that for any f in $L^2(D)$ and any z in D

$$(1) \qquad |f(z)| \leqq m_D(z)\|f\|_D,$$

(see, for example, [**1**, p. 5]).

Let ϕ_n be any complete orthonormal sequence in $L^2(D)$. Then from (1) follows

$$(2) \qquad \sum_{n=1}^{\infty} |\phi_n(z)|^2 \leqq [m_D(z)]^2,$$

and hence the convergence of the expansion for the Bergman reproducing kernel

$$(3) \qquad K_D(z, w) = \sum_{n=1}^{\infty} \phi_n(z)[\phi_n(w)]^-$$

where $[\phi_n(w)]^-$ indicates the complex conjugate of $[\phi_n(w)]$, (cf. [**1**, pp. 6 and 9]).

Let K be a given compact subset of D. Then we might wish to try to approximate the kernel function in K by using a finite series. Because of (2) and the fact that $m_D(z)$ is continuous, it is clear that given an $\epsilon > 0$ and a complete orthonormal sequence $\{\phi_n\}$, there exists an $N = N(D, K, \epsilon, \{\phi_n\})$ such that $m \geqq N$ and $z, w \in K$ implies

$$\left| K_D(z, w) - \sum_{n=1}^{m} \phi_n(z)[\phi_n(w)]^- \right| < \epsilon.$$

It might be asked whether or not N can be chosen independently of

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$\{\phi_n\}$, i.e., if we can find an upper bound to the number of terms required in the series of (3) to approximate $K_D(z, w)$ in K, independently of which orthonormal sequence $\{\phi_n\}$ is used.

In this note, we show that such an upper bound cannot exist. To do this, we first prove a result which the writer and several of those he has shown it to consider quite remarkable. If K is a compact subset of D, and $\epsilon > 0$ is given, then there exists a complete orthonormal sequence $\{\phi_n\}$ in $L^2(D)$ such that for all $z \in K$ and all n, $|\phi_n(z)| < \epsilon$.

The method of proof used is of some interest since it is one of relatively few examples of the use of doubly orthogonal functions.

THEOREM 1. *Let D be a bounded domain in the complex plane, $L^2(D)$ the Hilbert space of all analytic functions in D for which $\|f\|_D < \infty$, and K a compact subset of D. Let $\epsilon > 0$ be given. Then there exists a complete orthonormal sequence $\{\phi_n(z)\}$ in $L^2(D)$ such that $\|\phi_n\|_K < \epsilon$ for all n.*

PROOF. We may assume that K is the closure of a domain contained in D, for if not we merely enlarge K to K' satisfying this property.

Under these hypotheses we have the existence of a doubly orthogonal sequence of functions $\{\psi_n(z)\}$ (cf. [1, pp. 14–17]), that is, a complete orthonormal sequence in $L^2(D)$ satisfying

$$(\psi_n, \psi_m)_K = \lambda_n \delta_{nm}, \qquad \lambda_n \searrow 0. \tag{4}$$

Indeed, $\sum \lambda_n < \infty$, but we do not need this here. Since $\lambda_n \to 0$, we have

$$\frac{1}{m} \sum_{n=1}^{m} \lambda_n \to 0 \text{ as } m \to \infty.$$

Given $\epsilon > 0$, choose N such that

$$\frac{1}{2^N} \sum_{n=1}^{2^N} \lambda_n < \epsilon^2, \qquad \lambda_n < \epsilon^2 \text{ for } n > 2^N. \tag{5}$$

Let $E_N = (\epsilon_{ij})$ be a $2^N \times 2^N$ matrix consisting entirely of elements $\epsilon_{ij} = \pm 1$ such that any two rows are orthogonal. The existence of such matrices is of course well known. One is easily constructed inductively, setting

$$E_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad E_{n+1} = \begin{pmatrix} E_n & E_n \\ E_n & -E_n \end{pmatrix}.$$

Set

$$\phi_i = (1/2^N)^{1/2} \sum_{j=1}^{2^N} \epsilon_{ij}\psi_j, \qquad i = 1, 2, \cdots, 2^N.$$
$$= \psi_i, \qquad i > 2^N.$$

This set of functions is clearly orthonormal and complete in $L^2(D)$. From (4) and (5) we see that the desired conclusion, $\|\phi_n\|_K < \epsilon$, holds for $n > 2^N$, while if $n \leqq 2^N$, then

$$\|\phi_n\|_K^2 = \frac{1}{2^N} \sum_{j=1}^{2^N} \lambda_j < \epsilon^2.$$

Corollary 1. *If K is a compact subset of the bounded plane domain D and if $\epsilon > 0$ is given, then there exists a complete orthonormal sequence $\{\phi_n(z)\}$ in $L^2(D)$ such that for all $z \in K$ and all n*

$$|\phi_n(z)| < \epsilon.$$

Proof. Let G be a domain containing K and such that $\overline{G} \subset D$. Let

$$m = \max_{z \in K} m_G(z) \tag{6}$$

where $m_G(z)$ is defined as in (1).

From Theorem 1, construct an orthonormal sequence $\{\phi_n\}$ in $L^2(D)$ such that

$$\|\phi_n\|_G < \frac{\epsilon}{m}$$

for all n. Then because of (1), for any $z \in K$ and any n

$$|\phi_n(z)| \leqq m_G(z)\|\phi_n\|_G < \epsilon.$$

Corollary 2. *Let K be a compact subset of the bounded plane domain D. Let $0 < \epsilon <$ max $K_D(z, z)/2$ for $z \in K$. Then there exists no integer N such that*

$$\left| K_D(z, w) - \sum_{n=1}^{N} \phi_n(z)[\phi_n(w)]^- \right| < \epsilon$$

for all $z, w \in K$ and any complete orthonormal sequence $\{\phi_n\}$ in $L^2(D)$.

Proof. Let N be given. Choose z_0 in K such that $K_D(z_0, z_0) > 2\epsilon$. From Corollary 1, choose a complete orthonormal sequence $\{\phi_n\}$ such that $|\phi_n(z_0)|^2 < \epsilon/N$ for all n. But then

$$\left| K_D(z_0, z_0) - \sum_{n=1}^{N} \phi_n(z_0)[\phi_n(z_0)]^- \right| \geqq K_D(z_0, z_0) - \sum_{n=1}^{N} |\phi_n(z_0)|^2 > \epsilon.$$

Finally we note that Theorem 1 is not confined to the space $L^2(D)$. For the proof we require only the existence of a doubly orthogonal sequence ψ_n with $\lambda_n \to 0$. Thus, we have the result:

THEOREM 2. *Let* H_1 *and* H_2 *be two complete, separable Hilbert spaces and let* $J\colon H_2 \to H_1$ *be a linear mapping of* H_2 *into* H_1. *Suppose that* J *is completely continuous. Then, given* $\epsilon > 0$, *there exists in* H_2 *a complete orthonormal sequence* $\{\phi_n\}$ *such that* $\|J\phi_n\|_1 < \epsilon$ *for all* n.

PROOF. It is easily proved that if J is completely continuous, there exists a doubly orthogonal sequence $\{\psi_n\}$, complete in H_2, and satisfying

$$(\psi_n, \psi_m)_2 = \delta_{nm},$$
$$(J\psi_n, J\psi_m)_1 = \lambda_n \delta_{nm}, \qquad \lambda_n \searrow 0.$$

The proof of this theorem then proceeds exactly as that of Theorem 1.

BIBLIOGRAPHY

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