with mild restrictions on ϕ ensures that u/v satisfies the maximum principle; and this is the property which underlies the present analysis.

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University of California, Los Angeles

A NOTE ON LINEAR ORDINARY DIFFERENTIAL EQUATIONS

STEPHEN P. DILIBERTO[1]

Let

$$\frac{dx}{dt} = A(t)x, \tag{1}$$

where x is an n-column vector

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

and $A = (a_{ij}(t))$ where $a_{ij}(t)$ are continuous real valued functions of time $(-\infty < t < +\infty)$. Let $y^1(t), \cdots, y^n(t)$ be any n-linearly independent solutions of (1) defined for all t. Let $B^1(t), \cdots, B^n(t)$ be the n normal-orthogonal vectors obtained from the set $\{y^i\}$ by the Gram-Schmidt orthogonalization process. Let $B(t)$ be the orthogonal matrix whose jth column is $B^j(t)$, and introduce a new variable u (an n-column vector) defined by

$$x = B(t)u. \tag{2}$$

u satisfies the linear differential equation

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$$\frac{du}{dt} = C(t)u \tag{3}$$

where C is related to A and B by

$$C(t) = B^{-1}(t)A(t)B(t) - B^{-1}(t)\frac{d}{dt}B(t). \tag{4}$$

We have shown[2] that

$$c_{ij}(t) = 0 \qquad \text{if } i > j.$$

We propose to show that the c_{ij} satisfies certain simple formulas for $i \leqq j$, and these will imply that the c_{ij} are bounded if the a_{ij} are bounded. Our first proof[2] of this fact was unsatisfactory.

We reemploy the convention that if B is an n-square matrix, B_i will denote the ith row of B *and also the row vector determined by the ith row of* B; B^j will denote the jth column of B and also the column vector determined by the jth column of B. If E, F, and G are three matrices (n-square) and $E = FG$, then $E_i = F_iG$ and $E^j = FG^j$, where in the latter two formulas one has the appropriate vector-matrix and matrix-vector multiplication. E_i^j will denote the $(i-j)$th element of E, and if $E = FG$, then $E_i^j = F_iG^j$ where the right side is *scalar multiplication* (of a row vector times a column vector). If $E = (e_{ij})$, then $E_i^j = e_{ij}$.

From (4) one finds

$$c_{ij} = B_i^{-1}AB^j - B_i^{-1}\left(\frac{dB}{dt}\right)^j \tag{5}$$

or

$$c_{ij} = B'^iAB^j - B'^i\left(\frac{dB^j}{dt}\right), \tag{6}$$

this last following from the fact the B is orthogonal, i.e. B', the transpose of B, satisfies $B' = B^{-1}$ and therefore $(B')_i = B_i^{-1}$, but $(B')_i = (B^i)'$ or B'^i (in our notation). From $\delta_{ij} = B_i^{-1}B^j = B'^iB^j$, one finds on differentiating that

$$\left(\frac{d}{dt}B'^i\right)B^j = -B'^i\left(\frac{d}{dt}B^j\right) = -\left(\frac{dB'^j}{dt}\right)B^i. \tag{7}$$

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[2] S. P. Diliberto, *On systems of ordinary differential equations*, pp. 1–48 of *Contributions to the theory of non-linear oscillations*, Annals of Mathematics Studies, Princeton, 1950.

For $i=j$ (7) implies that

$$\left(\frac{d}{dt} B'^i\right) B^i = 0;$$

therefore (6) for $i=j$ becomes

$$c_{ii} = B'^i A B^i. \tag{8}$$

Formula (5) implies for $r>s$, $c_{rs}=0$; hence using (6) that

$$B'^r\left(\frac{dB^s}{dt}\right) = B'^r A B^s, \qquad r > s$$

and this combined with (7) implies

$$B'^s\left(\frac{dB^r}{dt}\right) = - B'^r A B^s \qquad (r > s). \tag{9}$$

For $s=i$ and $r=j$ and $i<j$ (9) substituted into (6) yields

$$c_{ij} = B'^i A B^j + B'^j A B^i. \tag{10}$$

Observing, when $A'=$transpose of A, that $B'^j A B^i = B'^i A' B^j$ one may rewrite (10) as

$$c_{ij} = B'^i (A + A') B^j. \tag{11}$$

Remarks. The fact that one has "explicit" formulas for c_{ij} (i.e. (5), (8), and (10)) does not appear to simplify our treatment (loc. cit.) of the theory of "generalized characteristic exponents." Formulas (8) and (10) can of course be used to establish a number of "stability" theorems; and all such results, including the formulas themselves, carry over directly to systems of linear ordinary differential equations in Hilbert space. It is to be noted that our expression for C does *not* depend on the derivatives of the $b_{ij}(t)$.

INSTITUTE FOR ADVANCED STUDY AND
UNIVERSITY OF CALIFORNIA, BERKELEY