APPROXIMATION OF CERTAIN FUNCTIONS BY
EXPONENTIALS ON A HALF LINE

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Introduction. In a recent paper, [1], A. Beurling has shown that
the positive translates of an integrable function defined on \([0, \infty)\)
generate, in a certain sense, at least one exponential of the form
e^{-iax}, x \geq 0, \text{ } a < 0,\text{ provided that the function does not vanish outside}
a finite interval. It is the converse problem with which we shall be
cconcerned here; namely, to what extent can the exponentials so gen-
erated be used to approximate the given function. We are able to give
what amounts to a complete solution.

The situation resembles strongly that of Schwartz' theory of mean
periodic functions [2]. M. Kahane has shown in [3] (see also [4])
how this theory can be presented very simply using the notion of
Fourier transform of a mean periodic function. Beurling also made
use of this method in the present case; however, we shall find it con-
venient to exploit this tool more systematically, in closer analogy
with Kahane's work. We shall also study our approximations in a
topology (the same as the one used in the theory of mean periodic
functions) which is simpler than that of Beurling.

Beurling based his work mainly on a certain division theorem
which states roughly that an entire function is of finite exponential
type if it is bounded on a half plane and equal to the ratio of two
bounded analytic functions on the complementary half plane. The
conclusions we make here follow from a refinement of this given in
\$3\$ which yields an upper bound for the type of such an entire func-
tion.

It should be remarked that B. Nyman ([7, pp. 28-29]) has estab-
lished a result similar to the one given here, using, however, a quite
different topology. (The referee calls attention to this in his report;
although I have since had the opportunity to consult Nyman's
work, it was not accessible to me in New York at the first writing of
this paper.)

1. We shall consider functions \(f \in L_1(-\infty, \infty)\) which are continu-
ous on \([0, \infty)\) and vanish on \((-\infty, 0)\). Together with such functions
\(f\) we also consider their positive translates \(f_h, f_h(x) = f(x+h), h \geq 0.\)
\(\mathcal{E}\) denotes the space of continuous functions on \([0, \infty),\) and \(\mathcal{E}_p\) that
subspace of \(\mathcal{E}\) consisting of functions which can be uniformly ap-

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proximated on compacta by linear combinations of the $f_h$ (restricted to \([0, \infty))$, $h \geq 0$.

1.1. **Definition.** $f$ is said to be mean periodic on \([0, \infty)\) if and only if $E_f$ is a proper subset of $E$.

A standard argument due to F. Riesz\(^1\) shows that $f$ is mean periodic on \([0, \infty)\) if and only if there exists a nonzero measure $\hat{m}$ of compact support in \([0, \infty)\) so that $\int_0^\infty f_h(x) d\hat{m}x = 0$, $h \geq 0$. If $m$ is any measure, we use systematically the notation:

$$d\hat{m}x = dm(-x), \quad \hat{m}^* = m.$$

Then, the above condition that $f$ be mean periodic on \([0, \infty)\) is that for some $m \neq 0$ of compact support in \((- \infty, 0]$, $m * f = 0$ on \([0, \infty)\).

(As is customary, we write $(m * f)(x) = \int_{-\infty}^x f(x - y) dy$; $m * f$ is the convolution of $m$ with $f$.)

1.2. **Definition.** For $f$ mean periodic on \([0, \infty)\), $L_f$ denotes the infimum of all $L > 0$ so that there exists a measure $m \neq 0$ having support in \([-L, 0]\) with $m * f = 0$ on \([0, \infty)\).

Evidently, if $L_f > 0$ and $0 < k < L_f$, any continuous function on \([0, k]\) can be uniformly approximated by linear combinations of the $f_h$, $h \geq 0$, while for $k > L_f$ this is no longer true. For $m \neq 0$ of compact support in \((- \infty, 0]\) such that $m * f = 0$ on \([0, \infty)\) let $g = m * f$. Then $g$ is also of compact support in \((- \infty, 0]\), vanishing outside the smallest interval containing the support of $m$. We indicate the Fourier transform of any measure or function by placing a circumflex over the symbol denoting it, e.g.

$$\hat{m}(\lambda) = \int_{-\infty}^\infty e^{i\lambda x} dm(x).$$

$\hat{g}(\lambda)$ and $\hat{m}(\lambda)$ are then entire functions of finite exponential type, so

$$j(\lambda) = \frac{\hat{g}(\lambda)}{\hat{m}(\lambda)}$$

possesses a meromorphic extension to the entire complex plane. Since

\(^1\) Let $F$ be any subspace of $E$ closed with respect to uniform convergence on compacta. If $g$ is a function in $E \sim F$, then there exists a measure $m$ of compact support in \([0, \infty)\) so that $\int f g dm \neq 0$ whilst $\int f u dm = 0$ for $u \in F$. (The converse of this is evident.) For there must be a $\delta > 0$ and a finite interval $J$ in \([0, \infty)\) such that for no $u \in F$ is $|g - u| < \delta$ on $J$. Let $B$ denote the Banach space of continuous functions on $J$ under the uniform norm, and $F'$ the closed subspace of $B$ generated by the restrictions of the members of $F$ to $J$. If $g'$ is the restriction of $g$ to $J$, $g' : eB \sim F'$. By the Hahn-Banach theorem, there must be a continuous functional $A$ on $B$ so that $Ag' \neq 0$, $Au' = 0$ for $u' \in F'$. By the Riesz representation theorem there is then a measure $m$ with support in $J$ so that $A v = \int v dm$, $v \in B$. This leads immediately to the desired conclusion.
Let \( f = 0 \) on \( (-\infty, 0) \) and \( f \in L_1 \), \( \hat{f}(\lambda) \) is regular and bounded for \( I\lambda \geq 0 \).

1.3. Theorem. A necessary and sufficient condition that the function \( x^n e^{-iax} \) \( (x \geq 0) \) be in \( E_f \) is that \( a \) be at least an \( n+1 \) fold pole of \( \hat{f}(\lambda) \).

The proof is the same as that of Theorem 1 in [3], but for the reader’s convenience, it is reproduced here.

Consider the case \( n = 0 \). Suppose \( a \) is a pole of \( \hat{f}(\lambda) \). By the above formula for \( \hat{f}(\lambda) \), we must have \( \hat{m}(a) = 0 \) for every measure \( m \) of compact support in \( (-\infty, 0] \) such that \( m * f = 0 \) on \([0, \infty) \). On restating this in terms of \( \hat{m} \) and using the result given in the footnote, we see that \( e^{-iax} \in E_f \).

If, conversely, \( e^{-iax} \in E_f \), pick a nonzero \( m \) of compact support in \( (-\infty, 0] \) so that \( m * f \) vanishes on \([0, \infty) \). We must then have \( \hat{m}(a) = 0 \), and we are done unless also \( \hat{g}(a) = 0 \). In that case, consider the functions \( M = m * K \), \( G = g * K \), where

\[
K(x) = \begin{cases} 
 1, & x < 0, \\
 0, & x \geq 0,
\end{cases}
\]

(The convolution of a function \( g \) with another function is defined the same way as that of a measure \( m \) with another function was defined above, save that we use \( g(x)dx \) instead of \( dm(x) \).)

Since \( m \) and \( g \) have compact supports in \( (-\infty, 0] \) and \( \hat{m}(a) = \hat{g}(a) = 0 \), we see that \( M \) and \( G \) have compact supports in \( (-\infty, 0) \), and it is legitimate (remembering that \( f = 0 \) on \( (-\infty, 0) \)) to apply the usual commutivity and associativity rules for convolutions to conclude \( G = (m * f) * K = (m * K) * f = M * f \). In particular, \( M * f = G \) vanishes on \([0, \infty) \), whence \( \hat{M}(a) = 0 \), using again the fact that \( e^{-iax} \in E_f \).

Now

\[
\hat{K}(\lambda) = \frac{i}{\lambda - a}, \quad I\lambda > Ia,
\]

whence by analytic continuation \( \hat{G}(\lambda) = i\hat{g}(\lambda)/(\lambda - a) \) for all \( \lambda \). Since \( \hat{f}(\lambda) = \hat{G}(\lambda)/\hat{M}(\lambda) \), we are done if the zero of \( \hat{g}(\lambda) \) at \( a \) is of order one. Otherwise, the argument may be repeated to attain the desired conclusion.

In case \( n > 0 \), one notes that if \( x^n e^{-iax} \in E_f \), then \( x^m e^{-iax} \in E_f \) for \( m = 0, 1, \cdots, n-1 \), this following immediately from the definition of \( E_f \). The theorem for the general case is then established in the same way as above.

From the above theorem and the fact that \( \hat{f}(\lambda) \) is regular and bounded in \( I\lambda \geq 0 \), it follows that we can have \( x^n e^{-iax} \in E_f \) only for \( Ia < 0 \).
1.4. Definition. $X_f$ denotes the set of functions ("exponential monomials") of the form $x^n e^{-iax}$ in $E_f$.

The question with which we are concerned in this paper is whether or not a given $f$ mean periodic on $[0, \infty)$ can be uniformly approximated on compact subsets of $[0, \infty)$ by linear combinations of members of $X_f$. It turns out that this is not true, but that $f_h$ can be so approximated for $h$ sufficiently large.

1.5. Definition. $A_f$ denotes the infimum of all $h \geq 0$ such that $f_h$ can be approximated in the way mentioned above.

Our problem reduces to the study of the relation between the numbers $A_f$ and $L_f$.

We need one last preliminary result:

1.6. Theorem. Let $f$ be mean periodic on $[0, \infty)$ and let $m \neq 0$ be a measure of compact support in $(-\infty, 0]$ so that $m \ast f = 0$ on $[0, \infty)$. Let $\hat{p}$ be any measure of compact support in $(-\infty, 0]$ and let

$$u(x) = \begin{cases} (\hat{p} \ast f)(x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then $u$ is mean periodic on $[0, \infty)$ and $m \ast u = 0$ there. Moreover, the set of poles of $u(\lambda)$ is the set of poles of $\hat{f}(\lambda)$ diminished by the set of zeros of $\hat{p}(\lambda)$ (taking into account multiplicities).

Proof. The first statement follows because $f$ vanishes off $[0, \infty)$ while $\hat{m}$ and $\hat{p}$ have their supports there. Again because of this, $u = p \ast f + a$ function of compact support, so $u(\lambda) = \hat{p}(\lambda)f(\lambda) + a$ entire function, and the second statement follows.

2. We shall prove our division theorem.

2.1. Lemma. Let $H(z) = \int_0^1 e^{-zt} dmt$, $m \neq 0$, and let, for $y \geq 0$, $M(y) = \sup_x |H(x + iy)|$. Then always $M(y) > 0$ and for any $\epsilon > 0$ there is an integer $k > 0$ so that $M(k)/M(k - 1) < e^{1+\epsilon}$.

Proof. That $M(y) > 0$ follows from $m \neq 0$ by a standard uniqueness theorem. If the second conclusion is false, then for some $\epsilon > 0$, $M(k) \geq M(0)e^{(1+\epsilon)k}$ for $k$ integers $\to \infty$, whence $\lim \sup_{y \to \infty} \log M(y)/y \geq 1 + \epsilon$, which contradicts the definition of $H(z)$.

2.2. Theorem. Let $F(z) = G(z)/H(z)$ be analytic everywhere, with $G(z)$ and $H(z)$ entire. Suppose

1. $|F(z)|$ is bounded for $Iz \geq 0$.
2. For $k > 0$, $|G(z)|$ is bounded for $Iz \leq k$.
3. $H(z) = \int_0^1 e^{-zt} dmt$, $m \neq 0$.

Then, for any $\epsilon > 0$ there exists a constant $K$ so that
\[ |F(z)| \leq Ke^{(A+\epsilon)|z|}. \]

**Proof.** By a change of variable, we may reduce the theorem to the case \( A = 1 \), which we suppose done. Retaining the notation of 2.1, choose an integer \( k > 0 \) so that \( \log \left( \frac{M(k)}{M(k-1)} \right) \leq 1 + \epsilon/4 \), which is possible by that lemma. By definition of \( M(y) \) we can then find a real \( c \) so that
\[
\ln \frac{M(k)}{|H(c + (k - 1)i)|} \leq 1 + \frac{\epsilon}{2}.
\]

By means of the transformation
\[
(1) \quad z \rightarrow w = \frac{i + (z - c - ik)}{i - (z - c - ik)}
\]
map \( Iz \leq k \) conformally onto \(|w| \leq 1 \) so that \( c + (k - 1)i \) goes into 0. \(|H(z)|\) is bounded for \( Iz \leq k \), hence by the Phragmén-Lindelöf theorem, \(|H(z)| \leq M(k)\) for \( Iz \leq k \). Put \( H(z)/M(k) = h(w) \), \( G(z) = g(w) \) for \( z \rightarrow w \) in the above mapping. We have, for \(|w| < 1\), \(|h(w)| \leq 1\) and by hypothesis \(|g(w)| \leq \) some number \( C \).

By a well-known theorem [5, p. 188] we can write \( g(w) = P_1(w)g_1(w) \), \( h(w) = P_2(w)f(w) \), where \( P_1(w) \) and \( P_2(w) \) are Blaschke products, \( g_1(w) \) and \( f(w) \) are free of zeros in \(|w| < 1\); moreover, \(|P_1(w)| \leq 1\), \(|P_2(w)| \leq 1\), \(|g_1(w)| \leq C\), and \(|f(w)| \leq 1\) there. Since \( g(w)/M(k)h(w) = F(z) \) is free of poles in \(|w| < 1\), some of the zeros of \( P_1(w) \) must cancel those of \( P_2(w) \), and we have \( P_1(w)/P_2(w) = P(w) \), another Blaschke product, \(|P(w)| \leq 1\) in \(|w| < 1\). We therefore have, for
\[
(2) \quad Iz < k, \quad |F(z)| \leq \frac{C}{M(k)|f(w)|}.
\]

Since \(|f(w)| \leq 1\) and \( f(w) \) is free of zeros in \(|w| < 1\), we can write [5, p. 197]:
\[
(3) \quad \ln |f(re^{i\phi})| = -\frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)d\mu\phi}{1 + r^2 - 2r\cos(\theta - \phi)}
\]
for \( \mu \) a positive measure on \([0, 2\pi]\). Therefore
\[
\log |h(0)| = \log |P_2(0)| + \log |f(0)| \leq -\frac{1}{2\pi} \int_0^{2\pi} d\mu\phi
\]
and
\[
\frac{1}{2\pi} \int_0^{2\pi} d\mu\phi \leq -\log |h(0)| = \log \left| \frac{M(k)}{H(c + (k - 1)i)} \right| \leq 1 + \frac{\epsilon}{2}.
\]
by choice of \( k \) and \( c \). This, combined with (3) yields

\[
(4) \quad -\log |f(w)| \leq \frac{4}{1 - |w|^2} \left(1 + \frac{\epsilon}{2}\right).
\]

For \( z = Re^{i\Phi}, \pi \leq \Phi \leq 2\pi \), we have, on substituting in (1),

\[
\frac{4}{1 - |w|^2} = \frac{(1 + k)^2 + c^2 - 2R((1 + k) \sin \Phi + c \cos \Phi) + R^2}{k - R \sin \Phi}.
\]

Since \( k > 0 \), this expression is

(i) \( \leq \text{const} + \frac{R}{\sin b}, \pi + b \leq \Phi \leq 2\pi - b, \) \( b \) any fixed number between 0 and \( \pi/2; \)

(ii) \( \leq \text{const} + B' R^2 \) for some \( B' \), \( \pi \leq \Phi \leq 2\pi \).

Combining this with (2) and (4) we find:

\[
|F(Re^{i\Phi})| \leq \text{const} \cdot \exp \left(\frac{1 + \epsilon/2)R}{\sin b}\right), \quad \pi + b \leq \Phi \leq 2\pi - b,
\]

\[
|F(Re^{i\Phi})| \leq \text{const} \cdot \exp BR^2, \quad \pi \leq \Phi \leq 2\pi.
\]

Now \( |F(z)| \) is by hypothesis bounded on the real axis, so, applying the Phragmén-Lindelöf theorem in each of the sectors \( \pi \leq \Phi \leq \pi + b, 2\pi - b \leq \Phi \leq 2\pi \), we find

\[
|F(z)| \leq \text{const} \cdot \exp \left(\frac{1 + \epsilon/2 |z|}{\sin b}\right)
\]

holds in the lower half plane. Making \( b \) close enough to \( \pi/2 \), we get

\[
|F(z)| \leq Ke^{\left(1+\epsilon\right)|z|},
\]

which holds also in the upper half plane, since by hypothesis \( |F(z)| \) is bounded there. Q.E.D.

3. We are now able to establish the main result:

3.1. Theorem. For \( f \) mean periodic on \([0, \infty)\), \( 0 \leq A_f \leq L_f \).

Proof. In order to show that \( f \) is uniformly approximable by linear combinations of members of \( X_f \) on compact subsets of \([0, \infty)\), it is enough to show that for any measure \( \tilde{\varphi} \) of compact support therein which satisfies

\[
\int_0^\infty x^ne^{-iax}d\tilde{\varphi}x = 0 \quad \text{for all functions } x^ne^{-iax} \text{ in } X_f,
\]

one has \( \int_0^\infty f_\epsilon(x)d\tilde{\varphi}x = 0 \). (See above footnote.) Supposing that \( \tilde{\varphi} \) is such a measure, define

\[
u(x) = \begin{cases} (\varphi \ast f)(x), & x \geq 0, \\ 0, & x < 0. \end{cases}
\]
We will be done if we show that \( u(x) \) vanishes for \( x > L_f \).

Let \( \tilde{m} \neq 0 \) be a measure on \([0, L_f + \epsilon]\) so that \( m \ast f = 0 \) on \([0, \infty)\).

By 1.6, \( m \ast u = v \) has compact support in \(( -\infty, 0)\) and our assumption on \( \tilde{f} \) means by 1.3 that \( \tilde{f}(\lambda) \) vanishes at all the poles of \( \tilde{f}(\lambda) \), so by 1.6 again, \( \tilde{u}(\lambda) \) is analytic everywhere. \( \tilde{u}(\lambda) = \tilde{v}(\lambda)/\tilde{m}(\lambda) \); \( \tilde{v}(\lambda) \) is obviously bounded for \( \Lambda \leq k, k > 0 \), and

\[
\tilde{m}(\lambda) = \int_0^{L_f + \epsilon} e^{-\lambda x} d\tilde{m} x,
\]

while \( \tilde{u}(\lambda) \) is bounded for \( \Lambda \geq 0 \) as in the remarks following 1.2. We may therefore apply 2.2 to conclude \( |\tilde{u}(\lambda)| \leq K \exp(L_f + 2\epsilon)|\lambda| \).

By an obvious modification of the Paley-Wiener theorem [6, p. 109], it follows that \( u(x) = 0, x \geq L_f + 2\epsilon \), and squeezing \( \epsilon \), we obtain the required result.

3.2. Corollary. If \( f \) is mean periodic on \([0, \infty)\) and \( X_f = \{0\} \), then \( f(x) = 0 \) for \( x > L_f \).

3.3. We show finally by means of two examples that the inequality \( 0 \leq A_f \leq L_f \) established in 3.1 cannot be improved.

Example 1. Let

\[
f(x) = \begin{cases} 
\sum_{n=-\infty}^{\infty} a_n e^{-nx} e^{inx}, & x \geq 0, \\
0, & x < 0,
\end{cases}
\]

where the \( a_n \) are chosen all \( > 0 \) so that their infinite sum converges. It is easy to see (e.g., on computing \( \tilde{f}(\lambda) \) and using 1.3) that \( X_f = \{e^{-x} e^{inx}\} \) whence \( A_f = 0 \). \( X_f \) is clearly uniformly complete on any interval \([0, k]\), \( k < 2\pi \), since for a given continuous \( q \) on such an interval we may approximate \( e^2 q(x) \) uniformly by linear combinations of the \( e^{inx} \) thereon. But for \( k \geq 2\pi \), this is no longer true; only those continuous functions \( q \) for which \( q(x + 2\pi) = e^{-2\pi} q(x) \) can be so approximated.

So here \( L_f = 2\pi \), whereas \( A_f = 0 \).

Example 2. Let

\[
f(x) = \begin{cases} 
e^{-x}, & x \geq 1, \\
e^{-1}, & 0 \leq x \leq 1, \\
0, & x < 0.
\end{cases}
\]

For any \( \epsilon > 0 \) we can clearly find a measure \( \tilde{m} \) on \([1, 1 + \epsilon]\) so that \( \int_1^{1+\epsilon} e^{-x} d\tilde{m} x = 0 \). If we extend \( \tilde{m} \) so as to be zero on \([0, 1]\) we will have \( \int_0^{1+\epsilon} f_h(x) d\tilde{m} x = 0, h \geq 0 \). Therefore \( L_f \leq 1 \). Clearly, \( X_f = \{e^{-x}\} \) and since
$f(x) = \text{const on } [0, 1]$, $A_f = 1$. So here $A_f \geq L_f$; i.e., in conjunction with 3.1, $A_f = L_f$. (It is also easy to show directly that $L_f = 1$.) In conclusion, I would like to express my thanks to Dr. Peter Lax, who introduced me to this subject, and with whom I have had many helpful discussions.

**References**


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**FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING—VII. A PARTIAL DIFFERENTIAL EQUATION FOR THE FREDHOLM RESOLVENT**

RICHARD BELLMAN

1. **Introduction.** Let $K(x, y)$ be a symmetric kernel over the square $0 \leq x, y \leq T$, continuous in both variables in this region, and possessing the additional property that $\int_0^T \int_0^T K(x, y)u(x)u(y)dx dy + \int_0^T u^2(x)dx$ is positive definite. Then the Fredholm integral equation

\[ u(x) + v(x) + \int_a^T K(x, y)u(y)dy = 0, \quad 0 \leq a \leq T, \]

has a unique solution for any function $v(x)$ continuous for $a \leq x \leq T$. This solution may be represented in the form

\[ u(x) = -v(x) + \int_a^T Q(x, y, a)v(y)dy. \]

Let us call the kernel $Q(x, y, a)$ the Fredholm resolvent.

The purpose of this note is to show that $Q(x, y, a)$ satisfies the