APPROXIMATION OF CERTAIN FUNCTIONS BY EXPONENTIALS ON A HALF LINE

PAUL KOOSIS

Introduction. In a recent paper, [1], A. Beurling has shown that the positive translates of an integrable function defined on $[0, \infty)$ generate, in a certain sense, at least one exponential of the form e^{-iax}, $x \geq 0$, $Ia < 0$, provided that the function does not vanish outside a finite interval. It is the converse problem with which we shall be concerned here; namely, to what extent can the exponentials so generated be used to approximate the given function. We are able to give what amounts to a complete solution.

The situation resembles strongly that of Schwartz' theory of mean periodic functions [2]. M. Kahane has shown in [3] (see also [4]) how this theory can be presented very simply using the notion of Fourier transform of a mean periodic function. Beurling also made use of this method in the present case; however, we shall find it convenient to exploit this tool more systematically, in closer analogy with Kahane's work. We shall also study our approximations in a topology (the same as the one used in the theory of mean periodic functions) which is simpler than that of Beurling.

Beurling based his work mainly on a certain division theorem which states roughly that an entire function is of finite exponential type if it is bounded on a half plane and equal to the ratio of two bounded analytic functions on the complementary half plane. The conclusions we make here follow from a refinement of this given in §3 which yields an upper bound for the type of such an entire function.

It should be remarked that B. Nyman ([7, pp. 28–29]) has established a result similar to the one given here, using, however, a quite different topology. (The referee calls attention to this in his report; although I have since had the opportunity to consult Nyman's work, it was not accessible to me in New York at the first writing of this paper.)

1. We shall consider functions $f \in L_1(-\infty, \infty)$ which are continuous on $[0, \infty)$ and vanish on $(-\infty, 0)$. Together with such functions f we also consider their positive translates f_h, $f_h(x) = f(x+h)$, $h \geq 0$. E denotes the space of continuous functions on $[0, \infty)$, and E_f that subspace of E consisting of functions which can be uniformly ap-

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proximated on compacta by linear combinations of the f_h (restricted to $[0, \infty)$), $h \geq 0$.

1.1. Definition. f is said to be mean periodic on $[0, \infty)$ if and only if E_f is a proper subset of E.

A standard argument due to F. Riesz[1] shows that f is mean periodic on $[0, \infty)$ if and only if there exists a nonzero measure $\check{m}$ of compact support in $[0, \infty)$ so that $\int_0^\infty f_h(x) d\check{m}x = 0$, $h \geq 0$. If m is any measure, we use systematically the notation:

$$d\check{m}x = dm(-x), \qquad \check{m}^{\check{}} = m.$$

Then, the above condition that f be mean periodic on $[0, \infty)$ is that for some $m \neq 0$ of compact support in $(-\infty, 0]$, $m * f = 0$ on $[0, \infty)$. (As is customary, we write $(m * f)(x) = \int_{-\infty}^{\infty} f(x-y) dmy$; $m * f$ is the convolution of m with f.)

1.2. Definition, For f mean periodic on $[0, \infty)$, L_f denotes the infimum of all $L > 0$ so that there exists a measure $m \neq 0$ having support in $[-L, 0]$ with $m * f = 0$ on $[0, \infty)$.

Evidently, if $L_f > 0$ and $0 < k < L_f$, any continuous function on $[0, k]$ can be uniformly approximated by linear combinations of the f_h, $h \geq 0$, while for $k > L_f$ this is no longer true. For $m \neq 0$ of compact support in $(-\infty, 0]$ such that $m * f = 0$ on $[0, \infty)$ let $g = m * f$. Then g is also of compact support in $(-\infty, 0]$, vanishing outside the smallest interval containing the support of m. We indicate the Fourier transform of any measure or function by placing a circumflex over the symbol denoting it, e.g.

$$\hat{m}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} dmx.$$

$\hat{g}(\lambda)$ and $\hat{m}(\lambda)$ are then entire functions of finite exponential type, so

$$\hat{f}(\lambda) = \frac{\hat{g}(\lambda)}{\hat{m}(\lambda)}$$

possesses a meromorphic extension to the entire complex plane. Since

[1] Let F be any subspace of E closed with respect to uniform convergence on compacta. If g is a function in $E \sim F$, then there exists a measure m of compact support in $[0, \infty)$ so that $\int g dm \neq 0$ whilst $\int u dm = 0$ for $u \in F$. (The converse of this is evident.) For there must be a $\delta > 0$ and a finite interval J in $[0, \infty)$ such that for no $u \in F$ is $|g-u| < \delta$ on J. Let B denote the Banach space of continuous functions on J under the uniform norm, and F' the closed subspace of B generated by the restrictions of the members of F to J. If g' is the restriction of g to J, $g' \epsilon B \sim F'$. By the Hahn-Banach theorem, there must be a continuous functional A on B so that $Ag' \neq 0$, $Au' = 0$ for $u' \epsilon F'$. By the Riesz representation theorem there is then a measure m with support in J so that $Av = \int v dm$, $v \epsilon B$. This leads immediately to the desired conclusion.

$f=0$ on $(-\infty, 0)$ and $f \in L_1$, $\hat{f}(\lambda)$ is regular and bounded for $I\lambda \geq 0$.

1.3. THEOREM. *A necessary and sufficient condition that the function* $x^n e^{-iax}$ $(x \geq 0)$ *be in* E_f *is that* a *be at least an* $n+1$ *fold pole of* $\hat{f}(\lambda)$.

The proof is the same as that of Theorem 1 in [**3**], but for the reader's convenience, it is reproduced here.

Consider the case $n=0$. Suppose a is a pole of $\hat{f}(\lambda)$. By the above formula for $\hat{f}(\lambda)$, we must have $\hat{m}(a)=0$ for every measure m of compact support in $(-\infty, 0]$ such that $m*f=0$ on $[0, \infty)$. On restating this in terms of $\hat{m}$ and using the result given in the footnote, we see that $e^{-iax} \in E_f$.

If, conversely, $e^{-iax} \in E_f$, pick a nonzero m of compact support in $(-\infty, 0]$ so that $m*f$ vanishes on $[0, \infty)$. We must then have $\hat{m}(a) =0$, and we are done unless also $\hat{g}(a)=0$. In that case, consider the functions $M=m*K$, $G=g*K$, where

$$K(x) = \begin{cases} e^{-iax}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

(The convolution of a function g with another function is defined the same way as that of a measure m with another function was defined above, save that we use $g(x)dx$ instead of dmx.)

Since m and g have compact supports in $(-\infty, 0]$ and $\hat{m}(a)=\hat{g}(a) =0$, we see that M and G have compact supports in $(-\infty, 0)$, and it is legitimate (remembering that $f=0$ on $(-\infty, 0)$) to apply the usual commutivity and associativity rules for convolutions to conclude $G=(m*f)*K=(m*K)*f=M*f$. In particular, $M*f=G$ vanishes on $[0, \infty)$, whence $\hat{M}(a)=0$, using again the fact that $e^{-iax} \in E_f$. Now

$$\hat{K}(\lambda) = \frac{i}{\lambda - a}, \qquad I\lambda > Ia,$$

whence by analytic continuation $\hat{G}(\lambda)=i\hat{g}(\lambda)/(\lambda-a)$ for all λ. Since $\hat{f}(\lambda)=\hat{G}(\lambda)/\hat{M}(\lambda)$, we are done if the zero of $\hat{g}(\lambda)$ at a is of order one. Otherwise, the argument may be repeated to attain the desired conclusion.

In case $n>0$, one notes that if $x^n e^{-iax} \in E_f$, then $x^m e^{-iax} \in E_f$ for $m=0, 1, \cdots, n-1$, this following immediately from the definition of E_f. The theorem for the general case is then established in the same way as above.

From the above theorem and the fact that $\hat{f}(\lambda)$ is regular and bounded in $I\lambda \geq 0$, it follows that we can have $x^n e^{-iax} \in E_f$ only for $Ia < 0$.

1.4. DEFINITION. X_f denotes the set of functions ("exponential monomials") of the form $x^n e^{-i\alpha x}$ in E_f.

The question with which we are concerned in this paper is whether or not a given f mean periodic on $[0, \infty)$ can be uniformly approximated on compact subsets of $[0, \infty)$ by linear combinations of members of X_f. It turns out that this is not true, but that f_h can be so approximated for h sufficiently large.

1.5. DEFINITION. A_f denotes the infimum of all $h \geq 0$ such that f_h can be approximated in the way mentioned above.

Our problem reduces to the study of the relation between the numbers A_f and L_f.

We need one last preliminary result:

1.6. THEOREM. *Let f be mean periodic on $[0, \infty)$ and let $m \neq 0$ be a measure of compact support in $(-\infty, 0]$ so that $m*f=0$ on $[0, \infty)$. Let p be any measure of compact support in $(-\infty, 0]$ and let*

$$u(x) = \begin{cases} (p*f)(x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then u is mean periodic on $[0, \infty)$ and $m*u=0$ there. Moreover, the set of poles of $\hat{u}(\lambda)$ is the set of poles of $\hat{f}(\lambda)$ diminished by the set of zeros of $\hat{p}(\lambda)$ (taking into account multiplicities).

PROOF. The first statement follows because f vanishes off $[0, \infty)$ while $\check{m}$ and $\check{p}$ have their supports there. Again because of this, $u=p*f+$a function of compact support, so $\hat{u}(\lambda)=\hat{p}(\lambda)\hat{f}(\lambda)+$an entire function, and the second statement follows.

2. We shall prove our division theorem.

2.1. LEMMA. *Let $H(z)=\int_0^1 e^{-izt}dmt$, $m \neq 0$, and let, for $y \geq 0$, $M(y) = \sup_x |H(x+iy)|$. Then always $M(y)>0$ and for any $\epsilon>0$ there is an integer $k>0$ so that $M(k)/M(k-1) \leq e^{1+\epsilon}$.*

PROOF. That $M(y)>0$ follows from $m \neq 0$ by a standard uniqueness theorem. If the second conclusion is false, then for some $\epsilon>0$, $M(k) \geqq M(0)e^{(1+\epsilon)k}$ for k integers $\to\infty$, whence $\lim \sup_{y\to\infty} \log M(y)/y \geqq 1+\epsilon$, which contradicts the definition of $H(z)$.

2.2. THEOREM. *Let $F(z)=G(z)/H(z)$ be analytic everywhere, with $G(z)$ and $H(z)$ entire. Suppose*

1. *$|F(z)|$ is bounded for $Iz \geq 0$.*
2. *For $k>0$, $|G(z)|$ is bounded for $Iz \leq k$.*
3. *$H(z)=\int_0^A e^{-izt}dmt$, $m \neq 0$.*

Then, for any $\epsilon>0$ there exists a constant K so that

$$|F(z)| \leq Ke^{(A+\epsilon)|z|}.$$

PROOF. By a change of variable, we may reduce the theorem to the case $A=1$, which we suppose done. Retaining the notation of 2.1, choose an integer $k>0$ so that $\log(M(k)/M(k-1)) \leq 1+\epsilon/4$, which is possible by that lemma. By definition of $M(y)$ we can then find a real c so that

$$\ln \frac{M(k)}{|H(c+(k-1)i)|} \leq 1 + \frac{\epsilon}{2}\cdot$$

By means of the transformation

$$z \to w = \frac{i+(z-c-ik)}{i-(z-c-ik)} \tag{1}$$

map $Iz \leq k$ conformally onto $|w| \leq 1$ so that $c+(k-1)i$ goes into 0. $|H(z)|$ is bounded for $Iz \leq k$, hence by the Phragmén-Lindelöf theorem, $|H(z)| \leq M(k)$ for $Iz \leq k$. Put $H(z)/M(k) = h(w)$, $G(z) = g(w)$ for $z \to w$ in the above mapping. We have, for $|w| < 1$, $|h(w)| \leq 1$ and by hypothesis $|g(w)| \leq$some number C.

By a well-known theorem [5, p. 188] we can write $g(w) = P_1(w)g_1(w)$, $h(w) = P_2(w)f(w)$, where $P_1(w)$ and $P_2(w)$ are Blaschke products, $g_1(w)$ and $f(w)$ are free of zeros in $|w| < 1$; moreover, $|P_1(w)| \leq 1$, $|P_2(w)| \leq 1$, $|g_1(w)| \leq C$, and $|f(w)| \leq 1$ there. Since $g(w)/M(k)h(w) = F(z)$ is free of poles in $|w| < 1$, some of the zeros of $P_1(w)$ must cancel those of $P_2(w)$, and we have $P_1(w)/P_2(w) = P(w)$, another Blaschke product, $|P(w)| \leq 1$ in $|w| < 1$. We therefore have, for

$$Iz < k, \qquad |F(z)| \leq \frac{C}{M(k)|f(w)|}. \tag{2}$$

Since $|f(w)| \leq 1$ and $f(w)$ is free of zeros in $|w| < 1$, we can write [5, p. 197]:

$$\ln |f(re^{i\theta})| = -\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)d\mu\phi}{1+r^2-2r\cos(\theta-\phi)} \tag{3}$$

for μ a positive measure on $[0, 2\pi]$. Therefore

$$\log |h(0)| = \log |P_2(0)| + \log |f(0)| \leq -\frac{1}{2\pi}\int_0^{2\pi} d\mu\phi$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} d\mu\phi \leq -\log |h(0)| = \log\left|\frac{M(k)}{H(c+(k-1)i)}\right| \leqq 1 + \frac{\epsilon}{2}$$

by choice of k and c. This, combined with (3) yields

$$-\log |f(w)| \leq \frac{4}{1-|w|^2}\left(1+\frac{\epsilon}{2}\right). \tag{4}$$

For $z=Re^{i\Phi}$, $\pi\leq\Phi\leq 2\pi$, we have, on substituting in (1),

$$\frac{4}{1-|w|^2} = \frac{(1+k)^2+c^2-2R((1+k)\sin\Phi + c\cos\Phi)+R^2}{k-R\sin\Phi}.$$

Since $k>0$, this expression is

(i) $\leq$const$+R/\sin b$, $\pi+b\leq\Phi\leq 2\pi-b$, b any fixed number between 0 and $\pi/2$;

(ii) $\leq$const$+B'R^2$ for some B', $\pi\leq\Phi\leq 2\pi$.

Combining this with (2) and (4) we find:

$$|F(Re^{i\Phi})| \leq \text{const}\cdot\exp\frac{(1+\epsilon/2)R}{\sin b}, \qquad \pi+b\leq\Phi\leq 2\pi-b,$$

$$|F(Re^{i\Phi})| \leq \text{const}\cdot\exp BR^2, \qquad \pi\leq\Phi\leq 2\pi.$$

Now $|F(z)|$ is by hypothesis bounded on the real axis, so, applying the Phragmén-Lindelöf theorem in each of the sectors $\pi\leq\Phi\leq\pi+b$, $2\pi-b\leq\Phi\leq 2\pi$, we find

$$|F(z)| \leq \text{const}\cdot\exp\frac{(1+\epsilon/2)|z|}{\sin b}$$

holds in the lower half plane. Making b close enough to $\pi/2$, we get $|F(z)|\leq Ke^{(1+\epsilon)|z|}$, which holds also in the upper half plane, since by hypothesis $|F(z)|$ is bounded there. Q.E.D.

3. We are now able to establish the main result:

3.1. THEOREM. *For f mean periodic on* $[0, \infty)$, $0\leq A_f\leq L_f$.

PROOF. In order to show that f_h is uniformly approximable by linear combinations of members of X_f on compact subsets of $[0, \infty)$, it is enough to show that for any measure $\tilde{p}$ of compact support therein which satisfies

$$\int_0^\infty x^n e^{-iax} d\tilde{p}x = 0 \text{ for all functious } x^n e^{-iax} \text{ in } X_f,$$

one has $\int_0^\infty f_h(x)d\tilde{p}x=0$. (See above footnote.) Supposing that $\tilde{p}$ is such a measure, define

$$u(x) = \begin{cases} (p*f)(x), & x\geq 0,\\ 0, & x<0.\end{cases}$$

We will be done if we show that $u(x)$ vanishes for $x>L_f$.

Let $\tilde{m}\neq 0$ be a measure on $[0, L_f+\epsilon]$ so that $m*f=0$ on $[0, \infty)$. By 1.6, $m*u=v$ has compact support in $(-\infty, 0]$ and our assumption on $\tilde{p}$ means by 1.3 that $\hat{p}(\lambda)$ vanishes at all the poles of $\hat{f}(\lambda)$, so by 1.6 again, $\hat{u}(\lambda)$ is analytic everywhere. $\hat{u}(\lambda)=\hat{v}(\lambda)/\hat{m}(\lambda)$; $\hat{v}(\lambda)$ is obviously bounded for $I\lambda\leq k$, $k>0$, and

$$\hat{m}(\lambda) = \int_0^{L_f+\epsilon} e^{-i\lambda x} d\tilde{m}x,$$

while $\hat{u}(\lambda)$ is bounded for $I\lambda\geq 0$ as in the remarks following 1.2. We may therefore apply 2.2 to conclude $|\hat{u}(\lambda)|\leq K \exp (L_f+2\epsilon)|\lambda|$.

By an obvious modification of the Paley-Wiener theorem [**6**, p. 109], it follows that $u(x)=0$, $x\geq L_f+2\epsilon$, and squeezing ϵ, we obtain the required result.

3.2. COROLLARY. *If f is mean periodic on $[0, \infty)$ and $X_f=\{0\}$, then $f(x)=0$ for $x>L_f$.*

3.3. We show finally by means of two examples that the inequality $0\leq A_f\leq L_f$ established in 3.1 cannot be improved.

EXAMPLE 1. Let

$$f(x) = \begin{cases} \sum_{-\infty}^{\infty} a_n e^{-x} e^{inx}, & x \geq 0, \\ 0, & x < 0, \end{cases}$$

where the a_n are chosen all >0 so that their infinite sum converges. It is easy to see (e.g., on computing $\hat{f}(\lambda)$ and using 1.3) that $X_f=\{e^{-x}e^{inx}\}$ whence $A_f=0$. X_f is clearly uniformly complete on any interval $[0, k]$, $k<2\pi$, since for a given continuous q on such an interval we may approximate $e^x q(x)$ uniformly by linear combinations of the e^{inx} thereon. But for $k\geq 2\pi$, this is no longer true; only those continuous functions q for which $q(x+2\pi)=e^{-2\pi}q(x)$ can be so approximated.

So here $L_f=2\pi$, whereas $A_f=0$.

EXAMPLE 2. Let

$$f(x) = \begin{cases} e^{-x}, & x \geq 1, \\ e^{-1}, & 0 \leq x \leq 1, \\ 0, & x < 0. \end{cases}$$

For any $\epsilon>0$ we can clearly find a measure $\tilde{m}$ on $[1, 1+\epsilon]$ so that $\int_1^{1+\epsilon} e^{-x}d\tilde{m}x=0$. If we extend $\tilde{m}$ so as to be zero on $[0, 1)$ we will have $\int_0^{1+\epsilon} f_h(x)d\tilde{m}x=0$, $h\geq 0$. Therefore $L_f\leq 1$. Clearly, $X_f=\{e^{-x}\}$ and since

$f(x)=\text{const}$ on $[0, 1]$, $A_f=1$. So here $A_f \geq L_f$; i.e., in conjunction with 3.1, $A_f=L_f$. (It is also easy to show directly that $L_f=1$.) In conclusion, I would like to express my thanks to Dr. Peter Lax, who introduced me to this subject, and with whom I have had many helpful discussions.

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NEW YORK UNIVERSITY

FUNCTIONAL EQUATIONS IN THE THEORY OF DYNAMIC PROGRAMMING—VII. A PARTIAL DIFFERENTIAL EQUATION FOR THE FREDHOLM RESOLVENT

RICHARD BELLMAN

1. **Introduction.** Let $K(x, y)$ be a symmetric kernel over the square $0 \leqq x, y \leqq T$, continuous in both variables in this region, and possessing the additional property that $\int_0^T \int_0^T K(x, y)u(x)u(y)dxdy + \int_0^T u^2(x)dx$ is positive definite. Then the Fredholm integral equation

$$u(x) + v(x) + \int_a^T K(x, y)u(y)dy = 0, \qquad 0 \leqq a \leqq T, \tag{1}$$

has a unique solution for any function $v(x)$ continuous for $a \leqq x \leqq T$. This solution may be represented in the form

$$u(x) = -v(x) + \int_a^T Q(x, y, a)v(y)dy. \tag{2}$$

Let us call the kernel $Q(x, y, a)$ the Fredholm resolvent.

The purpose of this note is to show that $Q(x, y, a)$ satisfies the

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