A PRÜFER TRANSFORMATION FOR MATRIX DIFFERENTIAL EQUATIONS

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Introduction. In the study of oscillation and boundedness of solutions of the second order scalar differential equation

$$(1)\qquad (p(x)y')' + f(x)y = 0, \qquad a \leqq x < \infty$$

the Prüfer polar-coordinate transformation

$$(2)\qquad y(x) = r(x)\sin\theta(x), \qquad p(x)y'(x) = r(x)\cos\theta(x)$$

has proved to be a useful tool [1].[2] In the present paper an analogous transformation is developed and applied to the corresponding second order self-adjoint (square) matrix differential equation

$$(3)\qquad (P(x)Y')' + F(x)Y = 0.$$

See [2; 4, Chap. IV; 6] for other discussions of this equation.

In order to pursue this analogy interpret the transformation (2) to be effected by expressing each nontrivial solution $y(x)$ of (1), and its corresponding function $p(x)y'(x)$, as a product of a positive function $r(x)$ and a solution of a second order differential equation of the form:

$$(4)\qquad (y'/q(x))' + q(x)y = 0, \text{ where } q(x) = \theta'(x), \text{ if } \theta'(x) \not\equiv 0;$$

or of the system:

$$(5)\qquad y' = q(x)z, \qquad z' = -q(x)y, \text{ if } \theta'(x) \text{ has zeros.}$$

The first section will be devoted to properties of solutions of the matrix system

$$(6)\qquad Y' = Q(x)Z, \qquad Z' = -Q(x)Y.$$

Then, in §2, it will be shown that for every nontrivial (matrix) solution $Y(x)$ of (3) there exists a nonsingular matrix $R(x)$, a symmetric matrix $Q(x)$ and solutions $Y=S(x)$, $Z=C(x)$ of (6) such that

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[1] These results were obtained while the author held a National Science Foundation grant, NSF-G1825 and was at Yale University. Now at the University of Utah.

[2] This paper contains a bibliography of the use of (2). To this list should be added paper [3] which appeared almost simultaneously with [1].

(7) $$Y(x) = S^*(x)R(x) \text{ and } P(x)Y'(x) = C^*(x)R(x)$$

where the (*) denotes the transpose of the indicated square matrix. It is to be noted that for $n=1$, (7) coincides with (2). Finally, in the last section nonoscillation theorems for (3) and (6) are given, which add to the collection in [6]. Boundedness results have been treated in [2].

1. **Matrix sines and cosines.** Let $Q(x)$ be an $n \times n$ symmetric matrix of continuous functions on $a \leqq x < \infty$. Then by elementary existence theory there exists a solution pair of $n \times n$ matrices

(8) $$Y = S(x) \equiv S[a, x; Q], \qquad Z = C(x) \equiv C[a, x; Q]$$

of (6) which satisfies the initial conditions

$$Y(a) = 0, \qquad Z(a) = E \text{ (the identity matrix)}.$$

Note that if $n=1$ (the scalar case), or if Q commutes with its integral $\int_a^x Q$ (e.g. $Q=$constant or a diagonal matrix), then $S = \sin \int_a^x Q$ and $C = \cos \int_a^x Q$. In general, S and C cannot be expressed as the usual sine and cosine infinite series, but they behave much like matrix functions of $\int_a^x Q$.

The following three observations are easily verified.

(i) $Y=C$, $Z=-S$ also form a solution pair of (6) with initial conditions $Y(a)=E$, $Z(a)=0$.

(ii) If $Q(x)$ is nonsingular, both $S(x)$ and $C(x)$ are solutions of $(Q^{-1}(x)Y')' + Q(x)Y = 0$.

(iii) $$\begin{aligned} S'(x) &= Q(x)C(x), & C'(x) &= -Q(x)S(x), \\ S(a) &= 0,\ S'(a) = Q(a), & C(a) &= E,\ C'(a) = 0., \end{aligned}$$

THEOREM 1.1. *For a given symmetric continuous matrix $Q(x)$ on $a \leqq x < \infty$ the following identities are true*:

(9) $$C^*C + S^*S = E, \qquad C^*S = S^*C,$$

(10) $$CC^* + SS^* = E, \qquad CS^* = SC^*.$$

PROOF. $(C^*C+S^*S)'=0$ and $(C^*S-S^*C)'=0$. Hence both combinations in parentheses are constant matrices and (9) is established by evaluating them at $x=a$. The identities (10) may be treated in a similar manner and, although the derivatives are not immediately zero, it is easy to show that $L=CC^*+SS^*$ and $M=CS^*-SC^*$ satisfy the system

$$L' = QM - MQ, \quad M' = -QL + LQ, \quad L(a) = E, \quad M(a) = 0,$$

whose only solution is $L=E$, $M=0$. However, (10) follows directly from (9), and conversely, as is seen by the next lemma. The proof is due to Professor S. Kakutani.

LEMMA 1.1. *If A and B are two $n \times n$ real matrices such that $A^*A+B^*B=E$ and $A^*B=B^*A$, then $AA^*+BB^*=E$ and AB^*+BA^*.*

PROOF OF THE LEMMA. Note that the hypothesis gives that the left inverse of $A+iB$ is its adjoint A^*-iB^*, where $i=(-1)^{1/2}$. Hence, A^*-iB^* is also the right inverse of $A+iB$, i.e.

$$E = (A + iB)(A^* - iB^*) = (AA^* + BB^*) + i(BA^* - AB^*),$$

and the lemma is proved.

The trace of the first identity of (9) yields

$$\text{(iv)} \qquad \|S\|^2 + \|C\|^2 \equiv n,$$

where $\|A\|$ denotes the square root of the sum of the squares of the elements of a matrix A, and that every element of S, or C, is bounded on $a \leqq x < \infty$.

Oscillation properties of S and C will be discussed in the last section.

2. **The Prüfer transformation.** Consider the general equation

$$(3) \qquad (P(x)Y')' + F(x)Y = 0, \qquad a \leqq x < \infty,$$

where $P(x)$ and $F(x)$ are $n \times n$ symmetric matrices of continuous functions and $P(x)$ is positive definite for $a \leqq x < \infty$. For each pair $U(x)$, $V(x)$ of solutions of (3) let the constant matrix

$$(11) \qquad W(U, V) = U^*PV' - U^{*\prime}PV$$

be called the *Wronskian*. See [2] for its properties.

THEOREM 2.1. *If $Y(x)$ is any nontrivial (matrix) solution of (3), with the above conditions on the coefficients, such that $Y(a)=0$, then on $a \leqq x < \infty$, $W(Y, Y)=0$ and there exists a symmetric continuous matrix $Q(x)$ and a nonsingular continuously differentiable matrix $R(x)$ such that*

$$(12) \quad Y(x) = S^*[a, x; Q]R(x) \text{ and } P(x)Y'(x) = C^*[a, x; Q]R(x).$$

Furthermore, $R(x)$ and $Q(x)$ satisfy the differential and "integral" equations

$$(13) \qquad R' = \{SP^{-1}C^* - CFS^*\}R, \qquad R(a) = P(a)Y'(a),$$

$$(14) \qquad Q = CP^{-1}C^* + SFS^*.$$

PROOF. Suppose that Q and R exist such that (12) is true. Differentiation and observation (iii) yield $P^{-1}C^*R = C^*QR + S^*R'$ and $-FS^*R = -S^*QR + C^*R$. By multiplying on the left of the first equation by S and the second by C, adding and employing identities (10) it is seen that (13) follows.

Similarly,

$$\{CP^{-1}C^* + SFS^*\}R = QR. \tag{14'}$$

Suppose that $R(a)$ is a singular matrix, then $|Y'(a)| = 0$ and tnere exists a nonzero constant (column) vector α such that $Y'(a)\alpha = 0$. Let $\beta(x) = Y(x)\alpha$, then $\beta(x)$ is the vector solution of the vector matrix equation $(P\beta')' + F\beta = 0$ satisfying $\beta(a) = \beta'(a) = 0$. Uniqueness gives that $\beta(x) \equiv Y(x)\alpha \equiv 0$ and $|Y(x)| \equiv 0$, $a \leqq x < \infty$. However, by [4, p. 9], n is the highest order of a zero of a nontrivial solution of (3), which gives a contradiction and, hence, $R(a)$ is nonsingular. Since $R(x)$ is a solution of the first order linear equation (13) then $R(x)$ is nonsingular for $a \leqq x < \infty$. Therefore the right multiplication of (14′) by R^{-1} gives (14). Note that $R(x)$ can be determined, as for the scalar case, $n = 1$, by the identity

$$R^*R = Y^*Y + Y^{*\prime}P^*PY'. \tag{15}$$

Also, if for $a < x < b$, $|Y(x)| \neq 0$ then

$$P(x)Y'(x)Y^{-1}(x) = S^{-1}[a, x; Q]C[a, x; Q]. \tag{16}$$

For $n = 1$, the right-hand side of (16) becomes $\cot \int_a^x Q$ and it is a simple matter to solve for Q in terms of the inverse cotangent. Lacking such a tool for arbitrary order n, the "integral" equation (14) will be solved by successive approximations.

LEMMA 2.1. *If $Q_1(x)$ and $Q_2(x)$ are symmetric continuous matrices on $a \leqq x < \infty$ and $S_i = S[a, x; Q_i]$, $C_i = C[a, x; Q_i]$, $(i = 1, 2)$, then*

$$\left.\begin{matrix} \|S_2 - S_1\| \\ \|C_2 - C_1\| \end{matrix}\right\} \leqq 2n \int_a^x \|Q_2 - Q_1\|, \qquad a \leqq x < \infty. \tag{17}$$

PROOF OF THE LEMMA. For each $i = 1, 2$, the system $C_i' = -Q_iS_i$, $S_i' = Q_iC_i$, $S_i(a) = 0$, $C_i(a) = E$, may be expressed as a single equation $T_i' = G_iT_i$ where T_i is the $2n \times n$ matrix

$$\begin{pmatrix} S_i \\ C_i \end{pmatrix} \text{ and } G_i = \begin{pmatrix} 0 & Q_i \\ -Q_i & 0 \end{pmatrix}.$$

Subtraction yields the nonhomogenous equation

$$(T_2 - T_1)' - G_1(T_2 - T_1) = (G_2 - G_1)T_2 \text{ and } T_2(a) - T_1(a) = 0.$$

Let

$$M(x) = \begin{pmatrix} C_1 & S_1 \\ -S_1 & C_1 \end{pmatrix},$$

then $M' - G_1 M = 0$ and $M(0) = E$ and, since G_1 is skew-symmetric then $\|M(x)\| = \|E\| = n^{1/2}$ and $\|M^{-1}(x)\| = \|E\| = n^{1/2}$. Use of the solution $M(x)$ of the homogeneous equation yields

$$T_2(x) - T_1(x) = M(x) \int_0^x M^{-1}(s)\{G_2(s) - G_1(s)\} T_2(s) ds$$

from which the desired inequalities of the lemma follow.

To return to the solution of (14) by successive approximations, let $Q_0(x)$ be any continuous, symmetric $n \times n$ matrix on $a \leqq x < \infty$ and for each non-negative integer n let

$$\begin{aligned} S_n &= S[a, x; Q_n], \qquad C_n = C[a, x; Q_n] \text{ and} \\ Q_{n+1}(x) &= C_n P^{-1} C_n^* + S_n F S_n^*. \end{aligned} \tag{18}$$

Since,

$$\begin{aligned} &Q_{n+1} - Q_n \\ &= (1/2)(C_n - C_{n-1})P^{-1}(C_n^* + C_{n-1}^*) + (1/2)(C_n + C_{n-1})P^{-1}(C_n^* - C_{n-1}^*) \\ &\quad + (1/2)(S_n - S_{n-1})F(S_n^* + S_{n-1}^*) + (1/2)(S_n + S_{n-1})F(S_n^* - S_{n-1}^*), \end{aligned}$$

then by Lemma 2.1 the following Lipschitz inequality is achieved

$$\begin{aligned} \|Q_{n+1}(x) - Q_n(x)\| \leqq 4n\{\|P^{-1}(x)\| + \|F(x)\|\} \int_a^x \|Q_n - Q_{n-1}\|, \\ a \leqq x < \infty. \end{aligned} \tag{19}$$

It is now a simple matter to follow the well-known techniques of successive approximations to show that $\lim_{n\to\infty} Q_n(x)$ exists (termwise) and is continuous on $a \leqq x < \infty$ and that the limit matrix $Q(x)$ satisfies (14). Once $Q(x)$ is obtained then $R(x)$ is obtained immediately from (13) and it remains only to show

LEMMA 2.2. *If $R(x)$ and $Q(x)$ are a solution pair of* (13, 14) *and $Z(x) = S^*[a, x; Q]R(x)$ then $P(x)Z'(x) = C^*[a, x; Q]R(x)$ and $Z(x)$ is a solution of* (3) *which satisfies $Z(a) = 0$, and $Z'(a) = Y'(a)$.*

PROOF OF THE LEMMA. By differentiating and applying (iii), (9), (10), (13) and (14): $Z(a) = 0$, $Z' = S^*R' + C^*QR = P^{-1}C^*R$, $Z'(a) = P^{-1}(a)R(a) = Y'(a)$ and $(PZ')' = C^*R' - S^*QR = -FZ$. This completes the proof of the lemma and of Theorem 2.1.

3. **Nonoscillation theorems.** Note that if $n=1$ and $Q(x)$ is continuous and positive (definite) for $a \leqq x < \infty$ and $\sin \int_a^x Q$ is nonoscillatory (for large x) then $\int_a^\infty Q < \infty$. Furthermore, if $\int_a^\infty Q < \pi/2$ then $\sin \int_a^x Q \neq 0$ and $\cos \int_a^x Q \neq 0$ for $a < x < \infty$. These facts will be generalized to analogous ones for arbitrary order n. Consider first equation (3) with the assumptions of the preceding section on the coefficients.

Definition. Equation (3) is said to be nonoscillatory (for large x) provided that there exists a number $b \geqq a$ and a solution $U(x)$ such that $W(U, U)=0$ and $|U(x)| \neq 0$ for $b \leqq x < \infty$. That this is equivalent to nonexistence of conjugate points for the corresponding matrix-vector equation is observed in [6, p. 314].

Theorem 3.1 (*Necessary condition for nonoscillation*). *If* (3), *with the assumptions of* §2 *on the coefficients, is nonoscillatory then there exists a* (*matrix*) *solution* $Z(x)$ *and a number* $c > a$ *such that* $Z(x)$ *is nonsingular for* $c \leqq x < \infty$, $W(Z, Z) = 0$ *and if* $(Z^*PZ)^{-1} = (k_{ij}(x))$, *then* $\int_c^\infty |k_{ij}| < \infty$, $i, j = 1, \cdots, n$.

Proof. Since (3) is nonoscillatory then there exists $b \geqq a$ and a solution $U(x)$ such that $W(U, U)=0$ and $|U(x)| \neq 0$, $b \leqq x < \infty$. By differentiation it is easily seen that

$$Z(x) = U(x)H(x), \quad \text{where} \quad H(x) = \int_b^x (U^*PU)^{-1} \tag{20}$$

is a solution of (3) such that $Z(b)=0$, $W(Z, Z)=0$ and, since $H(x)$ is positive definite for $b < x < \infty$, then $|Z(x)| = |U(x)|\,|H(x)| \neq 0$ for $b < x < \infty$. Also $W(V, Z)=E$. Since $H = U^{-1}Z$ is symmetric and positive definite then $T(x) = H^{-1} = Z^{-1}U$ has these same properties. Furthermore, $T' = -H^{-1}H'H^{-1} = -Z^{-1}P^{-1}Z^{*-1}$ and if $c > b$,

$$T(x) = T(c) - \int_c^x (Z^*PZ)^{-1}, \qquad c \leqq x < \infty.$$

Because $T(x)$ is positive definite on $c \leqq x < \infty$, then its trace, $\operatorname{tr} T(x) > 0$. Also $\operatorname{tr}[(Z^*PZ)^{-1}] > 0$ for $c \leqq x < \infty$. Hence $\int_c^x \operatorname{tr}[(Z^*PZ)^{-1}] < \operatorname{tr} T(c)$ and $\int_c^\infty \operatorname{tr}[(Z^*PZ)^{-1}] < \infty$. Furthermore, $\|(Z^*PZ)^{-1}\| \leqq \operatorname{tr}(Z^*PZ)^{-1}$ and the conclusion, $\int_c^\infty |k_{ij}|\, dx < \infty$, follows where the absolute value signs are redundant for $i=j$.

Corollary 3.1.1. *Additional conclusions to Theorem* 3.1 *are* $\int_c^\infty (\operatorname{tr} P)^{-1}\|Z\|^{-2} < \infty$ *and* $\int_c^\infty \operatorname{tr}(P^{-1})\|Z\|^{-2} < \infty$.

Proof. From Theorem 3.1, $\int_c^\infty \operatorname{tr}[(Z^*PZ)^{-1}] < \infty$. Since $P(x)$ is symmetric and positive definite there exists a nonsingular matrix

$N(x)$ such that $P(x)=N^*(x)N(x)$. Furthermore, $\|P\| \leqq \|N\|^2 = \operatorname{tr} P \leqq n\|P\|$ and

$$\operatorname{tr}[(Z^*PZ)^{-1}] = \operatorname{tr}[(ZN)^{-1}(ZN)^{*-1}] = \|(ZN)^{-1}\|^2 \geqq n\|NZ\|^{-2}$$
$$\geqq n\|N\|^{-2}\|Z\|^{-2} = n(\operatorname{tr} P)^{-1}\|Z\|^{-2}.$$

Let $G=(Z^*PZ)^{-1}$ then $ZGZ^*=P^{-1}$ and $\operatorname{tr} G \geqq \|G\| \geqq \|P^{-1}\|\|Z\|^{-2} \geqq (1/n) \operatorname{tr}(P^{-1})\|Z\|^{-2}$. By combining these inequalities the lemma is proved.

COROLLARY 3.1.2. *If in addition to* (3) *being nonoscillatory, all solutions of* (3) *are bounded*[3] *then* $\int^\infty \operatorname{tr}(P^{-1}) < \infty$ *and*

$$\int^\infty (\operatorname{tr} P)^{-1} < \infty. \tag{22}$$

PROOF. Follows immediately from Corollary 3.1.1.

An immediate result is a generalization of the first comment of this section:

COROLLARY 3.1.3. *If $Q(x)$ is continuous, symmetric and positive definite on $a \leqq x < \infty$ such that $\int^\infty \operatorname{tr} Q = \infty$, then there exists at least one number $b > a$ such that $|S[a, b; Q]| = 0$.*

PROOF. According to observation (ii) of the first section, S satisfies (3), where $P=Q^{-1}$ and $F=Q$. If $|S| \neq 0$ for $a<x<\infty$ then by definition $(Q^{-1}Y')'+QY=0$ is nonoscillatory. But by Corollary 3.1.2, $\int^\infty \operatorname{tr}(P^{-1}) = \int^\infty \operatorname{tr} Q < \infty$ which gives a contradiction. This section will be concluded by a *sufficient condition* for nonoscillation which is a generalization of the second remark at the beginning of the section.

THEOREM 3.2. *If $Q(x)$ is a continuous, symmetric and positive definite matrix on $a \leqq x < \infty$ such that $\int_a^\infty \operatorname{tr} Q < \pi/2n^{1/2}$, then $|S[a, x; Q]| \neq 0$ and $|C[a, x; Q]| \neq 0$ for $a < x < \infty$.*

PROOF. Suppose that $|C|=0$ for $x=b$ and b is the smallest such number $>a$. Let $K(x)=SC^{-1}$, $a \leqq x < b$, then $K(a)=0$, $K(x)$ is symmetric and

$$K' = KQK + Q \quad \text{for } a \leqq x < b. \tag{23}$$

This equation may be written in the form

$$K' - AK - KA^* = Q, \quad \text{where} \quad A = (1/2)KQ.$$

According to [5] the solution of the homogeneous part of $Y'-AY-YA^*=0$ is $Y=J(x)MJ^*(x)$, where M is a constant matrix and $J(x)$

[3] Sufficient conditions for boundedness are given in [2].

is the solution of $J' = AJ$, $J(a) = E$. By the variation of constants method the solution of the nonhomogeneous equation is

$$K(x) = J(x)\left\{\int_a^x J^{-1}(t)Q(t)J^{*-1}(t)dt\right\}J^*(x), \qquad a < x < b.$$

Therefore, $K(x)$ is positive definite on $a<x<b$. Now $S=KC$ and $C' = -QS = -QKC$ and by the determinant equality for first order linear matrix equations, $|C(x)| = \exp\{-\int_a^x \operatorname{tr}(QK)\}$. Since $|C(b)| = 0$, then $\int_a^b \operatorname{tr}(QK) = \infty$.

However, since Q and K are both positive definite on $a<x<b$ then $\operatorname{tr}(QK) \leqq \|Q\|\|K\| \leqq (\operatorname{tr} Q)(\operatorname{tr} K)$ and by taking the trace of (23), $\operatorname{tr} K' = \operatorname{tr}(KQK) + \operatorname{tr} Q \leqq \operatorname{tr} Q\{n(\operatorname{tr} K)^2+1\}$. Integration of this inequality yields, for $a<x<b$,

$$\operatorname{Arctan}\{n^{1/2} \operatorname{tr} K(x)\} \leqq n^{1/2}\int_a^x \operatorname{tr} Q < \pi/2$$

and $0 < \operatorname{tr} K(x) < K_0 = (1/n^{1/2}) \tan\{n^{1/2}\int_a^\infty Q\} < \infty$. Therefore:

$$\int_a^x \operatorname{tr}(QK) \leqq \int_a^x (\operatorname{tr} Q)(\operatorname{tr} K) \leqq K_0\int_a^x \operatorname{tr} Q \leqq K_0\pi/2(n)^{1/2} < \infty,$$

which contradicts the earlier conclusion that $\int_a^b \operatorname{tr}(QK) = \infty$. Now, $S(x) = C(x)\int_a^x C^{-1}QC^{*-1}$ and the theorem is proved.

COROLLARY 3.2.1. *If for equation* (3) *the coefficients* $P(x)$ *and* $F(x)$ *are both symmetric, positive definite and continuous on* $a \leqq x < \infty$ *such that* $\int_a^\infty \max\{\operatorname{tr} P^{-1}(x), \operatorname{tr} F(x)\} < \pi/2n^{1/2}$ *and* $Y(x)$ *is any nontrivial (matrix) solution of* (3) *such that* $Y(a) = 0$ *then both* $|Y(x)| \neq 0$ *and* $|Y'(x)| \neq 0$ *for* $a<x<\infty$.

PROOF. By Theorem 2.1, $Y(x) = S^*[a, x; Q]R(x)$, $P(x)Y'(x) = C^*[a, x; Q]R(x)$ where $R(x)$ is nonsingular and $Q = CP^{-1}C^* + SFS^*$ on $a \leqq x < \infty$. Since P^{-1} and F are positive definite and $\|C\|^2 + \|S\|^2 = n$, then $\operatorname{tr} Q(x) \leqq \max\{\operatorname{tr} P^{-1}(x), \operatorname{tr} F(x)\}$ on $a \leqq x < \infty$. Also if α is any nonzero constant (column) vector then, for each x on $a \leqq x < \infty$, $\alpha^*Q(x)\alpha = (C^*\alpha)^*P^{-1}(x)(C^*\alpha) + (S^*\alpha)^*F(x)(S^*\alpha)$. Suppose that for some $x = x_0$, $C^*[a, x_0; Q]\alpha = 0$ and $S^*[a, x_0; Q]\alpha = 0$. But by (10) $CC^* + SS^* = E$ and, hence $\alpha = 0$. Thus $\alpha^*Q(x)\alpha > 0$, for every vector $\alpha \neq 0$ and all x on $a \leqq x < \infty$, and $Q(x)$ satisfies the hypotheses of Theorem 3.2. Hence, the corollary follows immediately.

In conclusion, note that Theorem 3.2 follows from Corollary 3.2.1 making them equivalent.

BIBLIOGRAPHY

1. J. H. Barrett, *Behavior of solutions of second order self-adjoint differential equations*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 247–251.
2. ———, *Matrix systems of second order differential equations*, Portugaliae Mathematica, vol. 14 (1955) pp. 79–89.
3. R. A. Moore, *The behavior of a linear differential equation of second order*, Pacific J. Math. vol. 5 (1955) pp. 125–145.
4. M. Morse, *Calculus of variations in the large*, New York, 1934.
5. W. T. Reid, *A matrix differential equation of the Riccati type*, Amer. J. Math. vol. 68 (1946) pp. 237–246.
6. R. L. Sternberg, *Variational methods and non-oscillation theorems for systems of differential equations*, Duke Math. J. vol. 19 (1952) pp. 311–322.

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A NOTE ON THE LAW OF LARGE NUMBERS

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Let $\{X_k\}$, $k=1, 2, \cdots$, be a sequence of independent random variables with mean $EX_k=0$, $k=1, 2, \cdots$. Let $F_k(x)$ be the distribution function of X_k, $k=1, 2, \cdots$. We list certain conditions which may or may not obtain for such a sequence:

$$\text{(i)} \qquad \sum_{k=1}^{n} \int_{|x|\geq n} dF_k \to 0 \qquad \text{as } n \to \infty;$$

$$\text{(ii)} \qquad \frac{1}{n}\sum_{k=1}^{n} \int_{|x|<n} x\,dF_k \to 0 \qquad \text{as } n \to \infty;$$

$$\text{(iii)} \qquad \frac{1}{n^2}\sum_{k=1}^{n} \int_{|x|<n} x^2\,dF_k \to 0 \qquad \text{as } n \to \infty;$$

$$\text{(iv)} \qquad \frac{1}{n^2}\sum_{k=1}^{n} \left\{ \int_{|x|<n} x^2\,dF_k - \left(\int_{|x|<n} x\,dF_k \right)^2 \right\} \to 0 \qquad \text{as } n \to \infty.$$

Kolmogorov proved in [1] that (i), (ii) and (iv) together are necessary and sufficient conditions for the classical weak law of large numbers. In [1, Satz XI] the statement is also made (without proof) that (i), (ii) and (iii) together are necessary and sufficient conditions for the classical weak law. This statement has appeared more recently in various texts and monographs. We show that this statement is in-

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