REFLEXIVITY OF THE LENGTH FUNCTION

ISRAEL HALPERIN[1] AND W. A. J. LUXEMBURG[2]

1. As in [1], S will denote a space of points P with a countably additive, non-negative measure $\gamma(E)$; all sets and functions considered will be γ-measurable; e will always denote a set of *finite* measure; E will be called *purely infinite* if $\gamma(E)=\infty$ and $e\subset E$ implies $\gamma(e)=0$.

λ is a *length function* if $\lambda(u)$ is defined, with $0\leqq\lambda(u)\leqq\infty$, for every non-negative function $u=u(P)$, and satisfies:

(L1) $\lambda(u)=0$ if $u(P)=0$ for almost all P.

(L2) $\lambda(u)\leqq\lambda(v)$ if $u(P)\leqq v(P)$ for all P.

(L3) $\lambda(u+v)\leqq\lambda(u)+\lambda(v)$.

(L4) $\lambda(ku)=k\lambda(u)$ for all $k>0$.

(L5) $u_1(P)\leqq u_2(P)\leqq\cdots$ for all P implies $\lambda(\sup u_n)=\sup\lambda(u_n)$.

u_E will denote the restriction of u to E, i.e. $u_E(P)=u(P)$ if P is in E, $=0$ otherwise; $\lambda(E)$ means $\lambda(u)$ with $u(P)=1$ for P in E, $=0$ otherwise; u_N will denote min (u, N). λ will be called *continuous* if for every u,

(L6) $\lambda(u)=\sup_e\lambda(u_e)$.

λ^* will denote the *conjugate*:

$$\lambda^*(v)=\sup\left(\int uv\,d\gamma;\ \lambda(u)\leqq 1\right).$$

It is easily verified that λ^* is a length function. Of course, if the only u with $\lambda(u)\leqq 1$ has $\lambda(u)=0$, then $\lambda^*(v)=0$ for all v.

By definition,

$$(1.1)\quad \lambda^*(v)=\sup\left(\textstyle\sum_E \inf(uv \text{ on } E)\gamma(E);\ \lambda(u)\leqq 1, \text{ all } \textit{finite} \text{ collections of disjoint } E\right)$$

with the convention that $0\infty=0$. If $\lambda^*(v_E)$ happens to be 0 for every purely infinite E then the same value is obtained in (1.1) when the E are restricted to sets of *finite* measure; hence, in this case, $\lambda^*(v)=\sup_e\lambda^*(v_e)$. In particular, $\lambda^*(v)=\sup_e\lambda^*(v_e)$ if $\lambda^*(v)<\infty$ or if S has no purely infinite sets.

2. λ is called *reflexive* if $\lambda^{**}=\lambda$, i.e., if $\lambda^{**}(u)=\lambda(u)$ for all u. In [1] it was pointed out that: (i) $\lambda^{**}(u)\leqq\lambda(u)$, $\lambda^{***}=\lambda^*$ always, and (ii) $\lambda^{**}=\lambda$ if λ is a levelling length function.

Now we shall prove that for arbitrary length function λ and arbitrary u:

Received by the editors August 2, 1956.

[1] Canadian Government Overseas-Award Fellow, 1954–1955.

[2] Fellow of the National Research Council of Canada.

(2.1) $\lambda(u) \geqq \lambda^{**}(u) \geqq \sup_e \lambda(u_e)$ always,

(2.2) $\lambda^{**}(u) = \sup_e \lambda(u_e)$ if $\lambda^{**}(u) < \infty$ or if S has no purely infinite sets.

It follows that $\lambda^{**} = \lambda$ if λ is continuous, in particular whenever S is σ-finite.[3] We shall show by examples in §5 that $\lambda^{**} = \lambda$ does not always hold so that the L^λ spaces defined in [1] are more general than the Köthe spaces. A necessary and sufficient condition for the equality $\lambda^{**} = \lambda$ is given in (4.1) below.

3. For real-valued functions $f(P)$ we write $\lambda(f)$ to mean $\lambda(u)$ with $u(P) = |f(P)|$. L^2 denotes the real Euclidean, Hilbert or hyper-Hilbert space of all f with $\|f\| = (\int |f(P)|^2 d\gamma) < \infty$; U denotes the set of f in L^2 with $\lambda(f) \leqq 1$.

We shall prove (2.1), (2.2) by means of the following steps:

(3.1) If $u = \sup u_n$ with u_n increasing and if $\lambda^{**}(u_n) = \lambda(u_n)$ for all n, then $\lambda^{**}(u) = \lambda(u)$.

(3.2) If $u_n(P) \to u(P)$ for almost all P and if $\lambda(u_n) \leqq 1$ for all n, then $\lambda(u) \leqq 1$.

(3.3) If $f_n(P) \to f(P)$ for almost all P and if $\lambda(f_n) \leqq 1$ for all n, then $\lambda(f) \leqq 1$.

(3.4) If $\int |f - f_n|^2 d\gamma \to 0$ and if $\lambda(f_n) \leqq 1$ for all n, then $\lambda(f) \leqq 1$.

(3.5) U is a closed convex subset of L^2.

(3.6) If u is in L^2 and if $\lambda(u) < \infty$, then $\lambda^{**}(u) = \lambda(u)$.

(3.7) If $u = u_e$ for some e, then $\lambda^{**}(u) = \lambda(u)$.

PROOF OF (3.1) TO (3.5). (3.1) follows from (L5).

Let $v_n = \inf_m (u_{n+m})$ so that $v_n \leqq u_n$ and $u = \sup v_n$ with v_n increasing; then (3.2) follows from (L5).

(3.3) follows from (3.2) since $|f_n(P)| \to |f(P)|$ for almost all P.

(3.4) follows from (3.3) since, for a suitable subsequence, $f_n(P) \to f(P)$ for almost all P.

(3.5) follows from (3.4).

PROOF OF (3.6). By (3.1) we need consider only all u_N, i.e. we may suppose u is bounded.

If $\lambda^{**}(u) \neq \lambda(u)$ then $\lambda^{**}(u) < \lambda(u) < \infty$, and if we use $u/\lambda(u)$ in place of u we may suppose $\lambda(u) = 1$.

Choose any $\rho > 1$. Then ρu is not in U. Hence there is a closed hyperplane in L^2 which separates ρu from U,[4] i.e., for some h in L^2

---

[3] *Note by W. A. J. Luxemburg.* A proof of $\lambda^{**} = \lambda$ for the case S σ-finite was found by me after I had heard that the same result had been obtained by G. G. Lorentz. Lorentz' proof (unpublished) turned out to be quite different from mine. Professor Halperin simplified and refined my proof to obtain (2.1) and (2.2) (see [2, pp. 10, 11, 28]).

[4] For example, the set of all $(\rho u + f_0)/2 + f$ with $f \perp (\rho u - f_0)$, where f_0 is uniquely defined by the requirement: $\|\rho u - f_0\| = \inf (\|\rho u - g\|; g \text{ in } U)$.

and some real number c,

$$\int \rho u h d\gamma > c \geqq \sup\left(\int fhd\gamma; f \text{ in } U\right). \tag{3.8}$$

By replacing $h(P)$ by $|h(P)|$ in (3.8) we may suppose $h(P) \geqq 0$ for all P. We may clearly suppose also that $h(P)=0$ wherever $u(P)=0$ so that for every w, $\int hwd\gamma = \sup_N \int h \min(w, Nu)d\gamma$.

Let w be arbitrary with $\lambda(w) \leqq 1$; then for every N, $\min(w, Nu)$ is in U, and (3.8) implies

$$\int \rho u h d\gamma > \sup\left(\int whd\gamma; \lambda(w) \leqq 1\right) = \lambda^*(h). \tag{3.9}$$

It follows that $0 < \lambda^*(h) < \infty$; using $h/\lambda^*(h)$ in place of h in (3.9) we obtain: $\lambda^{**}(u) > 1/\rho$ for all $\rho > 1$. Hence, since $\lambda(u) = 1$, $\lambda^{**}(u) \geqq \lambda(u)$. Thus the assumption $\lambda^{**}(u) \neq \lambda(u)$ leads to a contradiction and (3.6) must hold.

Proof of (3.7). If $\lambda^{**}(u) \neq \lambda(u)$ then $\lambda^{**}(u) < \lambda(u)$ and, using (3.1), we may suppose u is bounded; then $u = u_e$ is in L^2 and (3.6) implies that $\lambda(u) = \infty$. Choose a maximal collection of disjoint E with $E \subset e$, $\gamma(E) > 0$ and $\lambda(u_E) < \infty$; with e replaced by $e - \sum(E)$ and u replaced by its restriction to $e - \sum(E)$, we will have the preceding statement together with: $\lambda(u_E) = \infty$ whenever $E \subset e$, $\gamma(E) > 0$ or equivalently: $\lambda(E) = \infty$ whenever $E \subset e$, $\gamma(E) > 0$.

It follows that $\lambda(v) = \infty$ whenever $v(P) \neq 0$ on a set $E \subset e$ with $\gamma(E) > 0$; hence $\lambda^*(e) = 0$, and $\lambda^{**}(u) \geqq \int Nud\gamma$ for every finite N, since $\lambda^*(N_e) = 0$.

But $\lambda(u) > 0$ implies $\int ud\gamma > 0$; hence $\lambda^{**}(u) = \infty$. This contradicts: $\lambda^{**}(u) < \lambda(u)$, so that (3.7) must hold.

Proof of (2.1). $\lambda^{**}(u) \geqq \sup_e \lambda^{**}(u_e) = \sup_e \lambda(u_e)$, using (3.7).

Proof of (2.2). This follows from (2.1) and the last sentence of §1 (applied to λ^{**} in place of λ^*).

4. We now prove:

(4.1) $\lambda^{**} = \lambda$ if and only if: for each u either (i) $\lambda(u) = \sup_e \lambda(u_e)$ or (ii) there exists a purely infinite set $E = E(u)$ such that $\lambda(u_F) = \infty$ for all $F \subset E$, $\gamma(F) > 0$.

Proof of (4.1). *Sufficiency*. (2.1) shows that $\lambda^{**}(u) = \lambda(u)$ if $\lambda(u) = \sup_e \lambda(u_e)$. On the other hand, if $E(u)$ exists as in (ii) then, as in the second paragraph of the proof of (3.7), $\lambda^*(E) = 0$; hence $\lambda^{**}(u) = \infty = \lambda(u)$.

Necessity. If $\lambda^{**}(u) = \lambda(u) < \infty$, then (2.2) shows that $\lambda(u)$

$=\sup_e \lambda(u_e)$. If $\lambda^{**}(u)=\lambda(u)=\infty$ and $\lambda(u)>\sup_e \lambda(u_e)$ then $\lambda^{**}(u) >\sup_e \lambda^{**}(u_e)$; then the sentence following (1.1) shows that $\lambda^{**}(u_G) >0$ for some purely infinite set G; thus for some v with $\lambda^*(v)\leqq 1$ $\int_G uvd\gamma>0$, implying $\int_G uvd\gamma=\infty$; then for some $\epsilon>0$, $\int_E uvd\gamma=\infty$ where $E=$set of P in G for which $v(P)\geqq\epsilon$. Now this E satisfies (ii) of (4.1).

5. Examples.

(5.1) $\lambda\neq\lambda^{**}$ *and* $\lambda^{**}(u)=0$ *for all* u: Let S consist of one point P with $\gamma(P)=\infty$, $\lambda(u)=u(P)$. Then $\lambda^*(v)=\infty$ if $v(P)\neq 0$, $=0$ if $v=0$. Hence $\lambda^{**}(u)=0$ for all u.

(5.2) $\lambda\neq\lambda^{**}$ *and* $\lambda^{**}(u)=\sup_e\lambda(u_e)$ *for all* u: Let S consist of a noncountable collection of indices α, let $\gamma(E)=$number of indices in E if this is finite, $=\infty$ otherwise; and for $u=(u_\alpha;\ \alpha\in S)$ let $\lambda(u) =\sup\ (u_\alpha)+\aleph_0-\sup\ (u_\alpha)$.[5] Then for $v=(v_\alpha)$: $\lambda^*(v)=\sum v_\alpha$; for all u, $\lambda^{**}(u)=\sup\ (u_\alpha)=\sup_e\lambda(u_e)$.

(5.3) $\lambda=\lambda^{**}$ *but* λ is not *continuous*: Let S consist of one point P with $\gamma(P)=\infty$, $\lambda(u)=\infty$ if $u\neq 0$. Then $\lambda^*(v)=0$ for all v; $\lambda^{**}(u) =\lambda(u)$ for all u, but $u_e=0$ for all e; $\lambda(u_e)=0$; $\lambda^{**}(u)\neq\sup_e\lambda(u_e)$ if $u(P)\neq 0$.

References

1. H. W. Ellis and Israel Halperin, *Function spaces determined by a levelling length function*, Canadian Journal of Mathematics vol. 5 (1953) pp. 576–592.
2. W. A. J. Luxemburg, *Banach function spaces*, Thesis, Technische Hogeschool te Delft, 1955.

Queen's University, Kingston, Ont.

[5] If M is a collection of non-negative real numbers, $\aleph_0-\sup\ (M)$ means inf (k; $0\leqq k\leqq\infty$ and at most a countable number of elements of $M\leqq k$).