

THE EFFECT OF MAPS UPON THE DIMENSION OF SUBSETS OF THE DOMAIN SPACE

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“Space filling curves” constitute simple examples of maps which raise the dimension of each closed subset of the domain space of positive dimension. That is, in its usual construction, a map f of a 1-cell X onto an m -cell Y is such that each closed subset of positive dimension, or what amounts to the same, each nondegenerate sub-continuum of X , has an image of dimension m .

In case the dimension of X is greater than 1, maps with such strong dimension raising properties do not always exist. For example, there does not exist a map of a 2-cell into a 3-cell such that the image of each nondegenerate sub-continuum is 3, or even ≥ 2 . This fact is contained in the results below, all of which give some bound to the dimension raising possibilities of maps.

It should be noticed that for $m > n$, there do exist maps (of the n -cell I^n onto the m -cell I^m) such that the dimension of all n -dimensional sets are raised. Thus $fg: I^n \rightarrow I^m$ is such a map, where g is the projection of I^n onto a 1-cell I^1 and f is as indicated in the first paragraph. Here, certain $(n-1)$ -dimensional sets are mapped by g and hence by fg into points. This can be slightly improved (I^i as before):

EXAMPLE. *There exists a map $F: I^n \rightarrow I^m$, for all $n, m > n$, such that the dimension of each n -dimensional set is increased by F and the dimension of no closed set is decreased by F .*

Let $f: I^1 \rightarrow I^{m-n+1}$ be as in the first paragraph and let f_i denote the i th coordinate map of f , for $i = 1, 2, \dots, m-n+1$. Define $F: I^n \rightarrow I^m$ by

$$F(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{n-1}, f_1(x_n), f_2(x_n), \dots, f_{m-n+1}(x_n)).$$

Then F is light as f is; light maps on compact sets cannot decrease dimension (see [1, p. 91 or Theorem C below). The image of each n -dimensional set is of dimension m , as F inherits from f the property of sending open sets into sets with nonempty interiors.

Throughout, f will be a map of a separable metric space X of dimension n into a separable metric space Y of dimension m .

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These first four results are corollaries to Theorems A, B, and C below:

(1) *If $m < 2n$, then f does not raise dimension of all closed subsets of dimension 1 ($n \geq 1$).*

(2) *If for some positive integer i , $n = 2i$ and $m = 2i + 1$, then there is a closed set of dimension i whose dimension is not raised.*

(3) *If f increases the dimension of all closed 1-dimensional sets, then f must at least double the dimension of all closed sets.*

(4) *If X is compact and $n = m$, then among those closed sets whose dimensions are not raised by f are sets of all dimensions $\leq n$.*

We first prove Theorems A, B, and C and a "remark" which is of a familiar nature; C is fairly well known and is included for the sake of completeness.

REMARK. *If Z is a subset of X of dimension i , then there is a closed set $X' \subset X - Z$, of dimension $n - i - 1$.*

PROOF. As X is n -dimensional, there exist closed sets C_1 and C_2 in X such that no set of dimension $\leq n - 2$ separates C_1 and C_2 and such that $C_1 \cdot C_2 = 0$. By Theorem "B" of [1, p. 34], there exists a closed set B_1 separating C_1 and C_2 such that $\dim Z \cdot B_1 \leq i - 1$. As B_1 separates C_1 and C_2 , $\dim B_1 \geq n - 1$. Repeating this process, we obtain a closed set $B_2 \subset B_1$ such that $\dim Z \cdot B_2 \leq i - 2$ and $\dim B_2 \geq n - 2$, a closed set B_3, \dots, B_{i+1} . Then $\dim Z \cdot B_{i+1} \leq -1$ and $\dim B_{i+1} \geq n - i - 1$, so that B_{i+1} is the required closed set X' .

THEOREM A. *There is a closed subset of X of dimension at least $(n - 1)/2$ whose image is of dimension at most $m/2$.*

PROOF. This is clear if $m = 0$. If $m > 0$, let p and q denote non-negative integers such that $p \leq m/2$, $q \leq m/2$, and $p + q = m - 1$. By the corollary to Theorem III 3, [1, p. 32], Y can be expressed as the sum of two sets, P of dimension p , and Q of dimension q . Though not noted in [1], the proof given shows that P can be taken to be an F_σ set. Then $f^{-1}(P)$ is an F_σ set, say $f^{-1}(P) = \bigcup_{i=1}^{\infty} F_i$, where each F_i is closed.

Case 1. For some i , $\dim F_i \geq (n - 1)/2$. Then F_i is a closed set whose image is of dimension $\leq \dim P \leq m/2$.

Case 2. For all i , $\dim F_i < (n - 1)/2$. Then $\dim f^{-1}(P) < (n - 1)/2$ (Sum Theorem, [1, p. 30]). Then by the remark above, there is a closed set $X' \subset X - f^{-1}(P) = f^{-1}(Q)$, of dimension $\geq n - (n - 1)/2 - 1 = (n - 1)/2$. As the image of X' lies in Q , it has dimension $\leq m/2$.

THEOREM B. *Either there is a closed 1-dimensional set in X whose image is at most 1-dimensional, or there is a closed $(n - 1)$ -dimensional set in X whose image is at most $(m - 2)$ -dimensional.*

PROOF. Let $Y = P + Q$, where P is a 1-dimensional F_σ set, Q is $(m - 2)$ -dimensional, and $P \cdot Q = 0$. As in the proof of Theorem A, if $\dim f^{-1}(P) \geq 1$, then $f^{-1}(P)$ contains a 1-dimensional closed set whose image, lying in P , is at most 1-dimensional. If $\dim f^{-1}(P) < 1$, that is, ≤ 0 , then $f^{-1}(Q)$ contains a closed set of dimension $\geq n - 1$, whose image, lying in Q , has dimension $\leq m - 2$.

THEOREM C. *If X is compact, then there is a closed set of dimension $\geq n - 1$ whose image is of dimension $\leq m - 1$.*

PROOF. X contains an n -dimensional Cantor manifold X' . Let $g = f|X'$. There exists a compact set $Q \subset g(X')$ of dimension $\leq m - 1$ which separates $g(X')$. But then $g^{-1}(Q)$ separates X' and is therefore of dimension $\geq n - 1$; $f^{-1}(Q)$ is closed and has an image of dimension $\leq m - 1$.

PROOF OF COROLLARIES 1-4. Theorem B implies (1) by induction. Theorem A applied to (2) yields a closed set M such that $\dim M \geq (2i - 1)/2$ and $\dim f(M) \leq (2i + 1)/2$. As these dimensions have to be integers, $\dim M \geq i \geq \dim f(M)$. Corollary 3 follows from (1). Theorem C implies (4) by induction.

It is to be noticed that Theorems A, B, and C give certain information about the closed subsets of X of intermediate dimension:

(5) *If for some i , $n \geq 2^i$ and $m < 2^{i+1}$, then among those closed sets whose dimensions are less than doubled by f , are sets of dimension 1, 2, 4, \dots , 2^i .*

(6) *If for some integers i and j , $i \geq j$, $n \geq 2^i$, and $m < 2^i + 2^j$, then there exists a closed set of dimension 2^{i-j} whose dimension is not raised.*

(7) *If X is compact and for some integers i and j , $i \geq j$, $n \geq 2^i$, and $m < 2^i + 2^j$, then among those continua in X whose dimensions are not raised, are continua of all dimensions $\leq 2^{i-j}$.*

PROOF OF COROLLARIES 5-7. Corollary (5) is obtained by applying Theorem A, i times, together with the fact that a closed set of dimension $\geq k$ contains a closed set of dimension $= k$. Similarly, Theorem A applied j times yields (6); (7) follows from (6) and Theorem C.

Corollaries (1), (2), (4), (6), and (7) may be considered as attempts to determine the largest integer $k = k(n, m)$ such that:

If $f: X \rightarrow Y$, then among those closed sets whose dimensions are not increased by f are sets of all dimensions $\leq k(n, m)$.

A tabulation of $k(n, m)$ follows, where the occurrence of "i?" means that the corresponding value of k is known only to be at least i .

Perhaps the most interesting portion of the problem of determining the integers $k(n, m)$ which remains, can be phrased as follows:

QUESTION. Is there a map f defined on a space X of dimension

$m \backslash n$	1	2	3	4	5	6	7	8	9	10	...	$2^i + 2^j = 1$
1	1	0	0	0	0	0	0	0	0	0		
2	2	2	1	0?	0?	0?	0?	0?	0?	0?		
3	3	3	3	1?	1?	0?	0?	0?	0?	0?		
4	4	4	4	4	2?	1?	1?	0?	0?	0?		
5	5	5	5	5	5	2?	1?	1?	1?	0?		
.												
.												
.												
2^i	2^i	2^i										2^{i-j}

$n \geq 2$ which raises the dimension of each closed subset of X of positive dimension? Could f be a map of I^3 into I^4 ?

If there is such a map, Corollary (3) shows that it is sufficient to consider only the 1-dimensional sets and hence, for compact X , only the nondegenerate continua.

BIBLIOGRAPHY

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COMMENTS ON OPEN HOMOMORPHISMS¹

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The present note consists of two comments on the openness of certain homomorphisms, between linear topological spaces (l.t.s.) in the first case and metric groups in the second. Both comments present only refinements on earlier results; the methods, however, are somewhat different and may be of some interest.

Let \mathfrak{A} be any class of closed convex circled sets defined in every

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