ON A QUESTION CONCERNING PARTITIONING RAISED BY R. H. BING

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R. H. Bing on p. 341 of [1] raises the following question:

Question. Does there exist a positive integer n such that the following result holds for each continuous curve M, each positive number ϵ, and each pair of mutually exclusive closed subsets H and K of M? If R is a finite subset of M such that each point of R belongs to an arc in M of diameter less than ϵ that intersects $H+K$, there are two collections A_H and A_K of arcs satisfying the following conditions: (a) Each element of A_H intersects H but not K and each element of A_K intersects K but not H nor any element of A_H. (b) Each element of R belongs to an element of A_H+A_K. (c) Each element of A_H+A_K is of diameter less than $n\epsilon$.

If for some integer n the answer is yes, then E. E. Moise's method of partitioning would be validated (see [2] and [3]). Also, a simple technique yielding an affirmative answer would allow a more direct proof of partitioning by Bing's method and could probably be used to advantage on other problems.

Bing [1] gives an example to show that the answer is no for $n=1$. The present paper gives an example to show that the answer is no for $n=2$.

In the example given below, a metric will be defined such that with this metric the given point set has the desired metric property. The example was originally considered with a homeomorphism to Euclidean three space, where the image had the desired property. The metric given here follows a suggestion of R. H. Bing.

EXAMPLE. The example is described at the top of the following page.

The metric. Consider the example as a finite graph G, the sum of a finite number of segments, $s_1, s_2, \cdots, s_n$, such that if s_i and s_j have a point in common, $s_i \cdot s_j$ is an end point of both s_i and s_j, and such that each element a_i is the sum of elements of $s_1, s_2, s_3, \cdots, s_n$. Let each element s_i be of length slightly less than ϵ, say $\epsilon-\delta$. Now if v_i, v_j are vertices of segments of G, set $d(v_i, v_j)=0$ if $i=j$, $d(v_i, v_j) =N(\epsilon-\delta)$ if $i\neq j$, where N is the minimum number of a_i's whose sum is a continuum containing v_i and v_j. If x, y belongs to the same a_i, let $d(x, y)=\min$ (distance x to y along the line segment, $\epsilon-\delta$).

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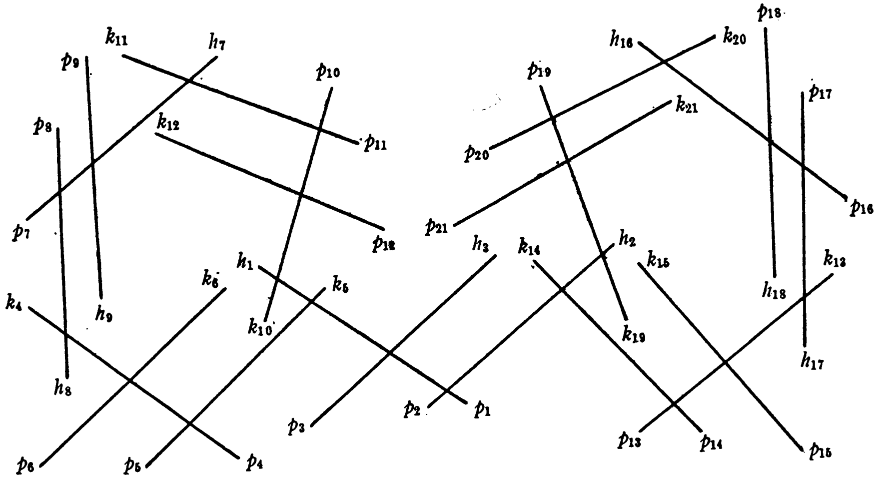
EXAMPLE



$$R = \sum_{i=1}^{21} p_i \qquad H = h_i(i = 1, 2, 3, 7, 8, 9, 16, 17, 18)$$
$$K = k_i(i = 4, 5, 6, 10, 11, 12, 13, 14, 15, 19, 20, 21)$$

a_i = the arc p_i to h_i (or k_i), $i = 1$ to 21. $G = \sum_{i=1}^{21} a_i$

Then for arbitrary x, y in G set $d(x, y) = \min (d(x, p) + d(p, q) + d(q, y))$ where minimum is taken over all p, q where $d(p, q)$ has already been defined.

Indication of a proof. Assume that n is two. Consider arcs a_1, a_2, a_3 (from p_1 to h_1, p_2 to h_2, and p_3 to h_3). Note that since $n=2$, p_3 must belong to an arc in A_H that is a subset of a_3 or a_3+a_1. Then p_1 belongs to an arc of A_H as does p_2. This means that either:

Case I. a_1 belongs to A_{H^*}; or,

Case II. a_2 belongs to A_{H^*}.

Suppose Case I. Then a_4 must belong to A_{K^*}, a_7 to A_{H^*}, and a_{10} to A_{K^*}, but a_{10} intersects a_1, which gives a contradiction. A similar line of reasoning leads to a contradiction for Case II.

REFERENCES

1. R. H. Bing, *Partitioning continuous curves*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 536–556.

2. E. E. Moise, *Grille decomposition and convexification theorems for compact metric continua*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 1111–1121.

3. ———. *A note of correction*, Proc. Amer. Math. Soc. vol. 2 (1951) p. 838.

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