INVERSIVE AND CONFORMAL CONVEXITY

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1. **Introduction.** A set S in Euclidean E_n—or, equivalently, on the n-sphere S_n—is called *inversively convex* if there is an inversion on an $(n-1)$-sphere in the space which transforms S into a convex set. A simple closed $(n-1)$-surface is inversively convex if one of the two regions bounded by it is inversively convex.

The group generated by the inversions can be characterized as follows [1, p. 102].

LEMMA 1. *Let T be a product of inversions of E_n. Then T is either a similarity or the product of an isometry and an inversion. The product of two concentric inversions is a similarity.*

PROOF. If T keeps the point at infinity fixed then it transforms the planes (spheres through infinity) into planes and is therefore affine. Since T is also conformal it is therefore a similarity.

Now let T transform infinity into the finite point p and let I_p be an inversion on a sphere with center p (*inversion at p* for short). Then I_pT keeps infinity fixed and is a similarity; or, by a suitable choice of radius for the sphere of I_p, an isometry U. Thus $T=I_pU$. The last statement of the lemma follows from the fact that such a product leaves infinity fixed.

As a consequence of Lemma 1 we see that inversive convexity is equivalent to convexity under an element of the group G generated by inversions. In the (complex) plane (Riemann sphere) G is the group of conformal and anticonformal linear fractional transformations. For $n>2$ the group G is the group of all conformal transformations [1, p. 100] and inversive convexity is therefore equivalent to *conformal convexity*. As another consequence of Lemma 1 we may now speak of *inversive convexity at p*, meaning convexity under inversion on the spheres with center p.

In §2 we give characterizations of inversive convexity of sets and curves in $E_2(S_2)$. Moreover, we characterize the centers of inversion which transform a given set into a convex set. In §3 new characterizations are given which are valid also for $n>2$ to yield a characterization of conformal convexity in $E_n(S_n)$.

2. **Inversive convexity in the plane.** A first simple characterization

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of inversive convexity at p, valid regardless of dimension, is given by the following.

THEOREM 1. *The set S is inversively convex at p if and only if for every two points $a, b \in S$ the arc ab of the circle determined by a, b, p which does not contain p, is contained in S. (If $p=a$ this reduces to the statement that p, b are joined by some circular arc in S.)*

The proof is obvious if we remember that the arc ab in question goes into the segment $a'b'$ of S', where x' indicates the image of x upon inversion at p.

From Theorem 1 we see that inversive convexity implies a certain kind of circle convexity. If S is inversively convex at two points p, q then we get a "lune convexity" from Theorem 1.

In order to obtain a more useful characterization we now pass to the question of inversive convexity of curves. A curve C in E_2 (S_2) is called *convex* if it lies in the boundary of its convex hull. A convex curve is thus either a simple arc (possibly with one endpoint at infinity) or a simple closed curve (possibly through infinity).

LEMMA 2. *An oriented simple curve C in the plane is convex if and only if all triangles $\triangle abc$ with vertices appearing in that order on C have the same orientation. (Collinear triples may be excluded or permitted according as we wish to define strict convexity or not.)*

PROOF. The interior of $\triangle abc$ lies in the convex hull $H(C)$ of C. Thus the orientation given to the boundary of $H(C)$ by that of C is the same as that of $\triangle abc$.

Conversely if C is not convex then the boundary of $H(C)$ contains at least three points a, b, c of C so that $\triangle abc$ contains in its interior a point $d \in C$. The four triangles determined by a, b, c, d can then not have the same orientation.

Now, on the simple, oriented curve C, each triple of points a, b, c determines a circle K_{abc}, whose orientation is given by the order given to a, b, c by the orientation of C. We label the open (closed) side of K_{abc} corresponding to one orientation of the plane by $S_o^+(a, b, c)$ (respectively $S_c^+(a, b, c)$) and the open (closed) side corresponding to the opposite orientation by $S_o^-(a, b, c)$ (respectively $S_c^-(a, b, c)$). We now define the intersections

$$S_o^+ = \bigcap_{a,b,c \in C} S_o^+(a, b, c); \qquad S_c^+ = \bigcap_{a,b,c \in C} S_c^+(a, b, c),$$

$$S_o^- = \bigcap_{a,b,c \in C} S_o^-(a, b, c); \qquad S_c^- = \bigcap_{a,b,c \in C} S_c^-(a, b, c).$$

Since inversion at a point p preserves the orientation of every circle containing p in its interior and reverses the orientation of every circle containing p in its exterior, Lemma 2 yields the following.

THEOREM 2. *A simple, oriented curve C becomes convex upon inversion at p if and only if* $p \in S_c^+ \cup S_c^-$, *and becomes strictly convex if and only if* $p \in S_o^+ \cup S_o^-$.

PROOF. From our construction we see that a point p lies in S_c^+ or S_c^- if and only if it lies in the closed interior of all K_{abc} with one orientation of $\triangle abc$ and in the closed exterior of all K_{abc} with the opposite orientation of $\triangle abc$. Thus by the remark preceding the theorem, upon inversion at p the image triples which are not collinear will be triangles which all have the same orientation. Thus according to Lemma 2 the image C' is convex. If $p \in S_o^+ \cup S_o^-$ then no triple a, b, c lies on a circle through p. Thus collinearity of a', b', c' is excluded and C' is strictly convex.

Conversely if p lies on opposite sides of K_{abc} and K_{def} then the images $\triangle a'b'c'$ and $\triangle d'e'f'$ upon inversion at p have opposite orientation and C' is not convex. Finally if $p \in (S_c^+ - S_o^+) \cup (S_c^- - S_o^-)$ then p lies on one of the circles K_{abc}. Thus the image C' contains collinear a', b', c' and is not strictly convex.

COROLLARY. *Let S be the exterior of a closed convex curve C* (*possibly through infinity*) *and P the set of points at which S is inversively convex, then P is a convex* (*possibly empty*) *closed subset of the closed interior of C.*

Instead of convexity we now consider the concept of *local convexity* of a curve.

DEFINITION. An oriented curve C is *locally convex* if every point x on C is contained in an open convex arc C_x of C so that the convex hulls $H(C_x)$ lie on the same side of C. A curve C is *inversively locally convex* if its image upon inversion on some circle is locally convex.

If the C_x are strictly convex then the condition on $H(C_x)$ becomes superfluous.

A locally convex curve need not be convex and need not even be simple. However a simple closed curve (possibly through infinity) which is locally convex is convex. Thus for simple closed curves the characterization to be given of inversive local convexity is a characterization of inversive convexity.

DEFINITION. A *limit circle* of a curve C at a point $x \in C$ is the limit of a sequence of circles K_{abc} with $a, b, c \in C$ as the arc a, b, c shrinks to the point x. If C is oriented then the approximating circles are oriented and we obtain *oriented limit circles* (possibly of zero radius).

LEMMA 3. *An oriented curve C is locally convex if and only if all its limit circles have the same orientation.*

PROOF. Each point $x\in C$ is contained in a convex subarc $C_x\subset C$. Hence by Lemma 2 all K_{abc} with $a, b, c\in C_x$ have the same orientation, which is also the orientation of any limit circle at x. Now let x, y be two points on C. By hypothesis C_x and C_y have the same orientation relative to their convex hulls. Hence the limit circles at x and y have the same orientation.

Assume now that C is not locally convex. Then either (i) there exists a point $x\in C$ so that every open subarc $C_x\subset C$ which contains x, contains points a, b, c so that x is an interior point of Δabc; or (ii) C contains a straight line segment xy so that $H(C_x)$, $H(C_y)$ are on opposite sides of xy. In case (i) not all the circles determined by three of the four points a, b, c, x can have the same orientation. If we let C_x shrink to x we thus obtain limit circles at x with opposite orientations. In case (ii) the limit circles of C_x and of C_y are on opposite sides of C.

THEOREM 3. *Let C be an oriented curve. Let $\mathcal{S}_c^+$ be the set of points which lie on one closed side of all the limit circles of C and $\mathcal{S}_c^-$ the set of points lying on the other closed side of all the limit circles of C. Then C is locally convex upon inversion at p if and only if $p\in\mathcal{S}_c^+\cup\mathcal{S}_c^-$.*

The derivation of Theorem 3 from Lemma 3 is entirely analogous to that of Theorem 2 from Lemma 2 and is therefore omitted.

If C is differentiable the limit circles are the circles of curvature of C, and Theorem 3 could then be stated in terms of circles of curvature. Another interesting special case is the following.

COROLLARY. *If C is a closed convex curve (possibly through infinity), then the intersection S of the closed interiors of the circles K_{abc} $(a, b, c\in C)$ is also the intersection of the closed interiors of the limit circles of C. (The "interior" of a circle of infinite radius is that half plane which contains C.) The set S is the set of points at which the exterior of C is inversively convex.*

If the curvature of C is bounded away from zero—or, more generally, if the limit circles of C have bounded radius, then the interior of C is inversively convex at every point sufficiently far from C.

3. **Conformal convexity.** Instead of extending the method of §2 to higher dimensions—which is possible in part but cumbersome—we introduce a new method which gives new insight also in the two-dimensional case.

DEFINITION. A *sphere of support* of a set S at a point x in the closure

$\bar{S}$ of S is a sphere K such that S lies entirely on one closed side of K. An *extremal sphere of support* of S at x is a sphere of support of S at x such that the side containing S (we shall call it the *positive side*) is minimal.

An extremal sphere of support may be a plane or degenerate to a single point. There may be several extremal spheres of support at a point.

Now we associate to each boundary point x of S the set K_x consisting of the union of the closed negative sides of the extremal spheres of support of S at x. Finally we let K_S be the intersection of the K_x as x runs through all boundary points of S.

LEMMA 4. *A set S with interior points is inversively convex at p if and only if there is a supporting sphere through p at every boundary point of S.*

PROOF. The image S' of S upon inversion at p is convex if and only if there is a plane of support to S' at every boundary point of S'. Inverting again we transform the planes into spheres of support of S through p.

THEOREM 4. *A set S with interior points is inversively convex at p if and only if $p \in K_S$. Thus inversive convexity of S is equivalent to the nonemptiness of K_S.*

PROOF. If S is inversively convex at p then according to Lemma 4 there is a sphere of support to S through p at every boundary point x of S. This sphere is contained in the closed negative side of some extremal sphere of support to S at x and hence $p \in K_x$ for every boundary point x, that is $p \in K_S$.

Conversely, assume $p \in K_S$, then for every boundary point x of S there is an extremal sphere of support K which contains p in its closed negative side. The sphere through p and tangent to K at x is therefore in the closed negative side of K and is a sphere of support to S at x (in case $x = p$ this choice is not unique). Thus, according to Lemma 4, S is inversively convex at p.

A combination of Theorem 4 with the Corollary to Theorem 3 yields some information about convex curves, (the analogous theorem also holds in higher dimensions).

COROLLARY. *For a closed convex twice differentiable curve C in the plane the following sets are identical:*

(i) *The intersection of the closed interiors of the circles of curvature of C, and the intersection of the closed interiors of the maximal inscribed circles to C.*

(ii) *The intersection of the closed exteriors of the circles of curvature of C, and the intersection of the closed exteriors of the minimal circumscribed circles to C.*

Reference

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HOMOTOPY GROUPS OF ONE-DIMENSIONAL SPACES

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In this paper we prove the following theorem:

If S is a one-dimensional separable metric space, then $\pi_k(S)=0$ *for all* $k>1$.

Actually it is proved that a much broader class of spaces than spheres have the property that mappings of these spaces into one-dimensional spaces are homotopic to constant maps. This class of spaces includes, for example, projective spaces and Lens spaces.

Lemma 1.[2] *Let X be a compact metric space whose one-dimensional integral singular homology group is a torsion group. Then for any finite covering G of order one by arcwise-connected open sets, G does not contain a simple loop.*

Proof. By a simple loop we mean a simple chain such that the first and last sets are the same. Let K be the nerve of G. Since K is one-dimensional, a simple loop in G implies a nonbounding one cycle in K. Hence it suffices to show that $H_1(K)=0$.

Let $\phi\colon X\rightarrow K$ be a canonical map. For each vertex v in K we choose a point $\psi(v)$ in the element of G corresponding to v. For each edge σ with vertices v_1 and v_2 we extend ψ on $\{v_1, v_2\}$ to a mapping of σ into the union of the two elements of G corresponding to v_1 and v_2. This is possible, since these two elements of G are arcwise connected and

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