THE L_p CONVERGENCE OF FOURIER-BESSEL SERIES FOR $0<p<1$

CHARLES STANDISH[1]

The object of this note is to extend to Fourier-Bessel series a theorem of M. Riesz on L_p convergence of trigonometric series. We prove the following

THEOREM. *If* $f(x)\in L(0, 1)$ *then*

$$\lim_{N\to\infty} \int_0^1 |f(x) - S_N(x)|^p dx = 0, \qquad 0 < p < 1,$$

where $\nu \geqq -1/2$,

$$S_N(x) = \sum_1^N b_n (2x)^{1/2} J_\nu(u_n x)/J_{\nu+1}(u_n),$$

$$J_{\nu+1}(u_n) b_n = 2^{1/2} \int_0^1 t^{1/2} J_\nu(u_n t) f(t)\, dt,$$

and $\{u_n\}$ *is the sequence of positive roots of* $J_\nu(x)$.

In the proof of the theorem we need the

LEMMA. *If* $f(x)\in L(a, b)$ *then*

$$\hat{f}(x) = \int_a^b f(t)(x-t)^{-1} dt$$

exists, is in L_p $(0<p<1)$ *and*

$$\int_a^b |\hat{f}(x)|^p dx \leqq A \left(\int_a^b |f(x)|\, dx \right)^p$$

where A is independent of $f(x)$.

This lemma is an easy consequence of a result of Loomis[2] [1, pp. 1085–1086].

We now proceed to the proof of the theorem. By the work of Wing [2, pp. 794–795],

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\begin{aligned}
|S_N(x)| \leqq & \int_0^1 |f(t)|/(x+t)dt + \int_0^1 |f(t)|/(2-x-t)dt \\
& + \frac{1}{2}\left|\int_0^1 (A_Nx)^{1/2}J_\nu(A_Nx)(A_Nt)^{1/2}J_{\nu+1}(A_Nt)f(t)/(x-t)dt\right| \\
& + \frac{1}{2}\left|\int_0^1 (A_Nx)^{1/2}J_{\nu+1}(A_Nx)(A_Nt)^{1/2}J_\nu(A_Nt)f(t)/(x-t)dt\right|
\end{aligned}
$$

where $A_N=(N+\nu/2+1/4)\pi$. Since $(A_Nx)^{1/2}J_\nu(A_Nx)$ is bounded in N and x and $(A_Nt)^{1/2}J_{\nu+1}(A_Nt)$ is bounded in N and t [**2**, p. 796], by using the inequality

$$\int |f(x)+g(x)|^p dx \leqq \int |f(x)|^p dx + \int |g(x)|^p dx \quad (0<p<1)$$

we obtain

$$\int_0^1 |S_N(x)|^p dx \leqq (\text{constant}) \left(\int_0^1 |f(x)|\, dx\right)^p.$$

By classical results on the convergence of Fourier-Bessel series [**4**, pp. 593–594, 612] given $\epsilon>0$ we can obtain a linear combination of functions $\{(2x)^{1/2}J_\nu(u_kx)\}$ such that

$$\int_0^1 \left| f(x) - \sum_1^N c_{k_\epsilon}(2x)^{1/2}J_\nu(u_kx)\right| dx < \epsilon$$

and

$$\int_0^1 \left| f(x) - \sum_1^N c_{k_\epsilon}(2x)^{1/2}J_\nu(u_kx)\right|^p dx < \epsilon.$$

The remainder of the proof proceeds as in the case of ordinary Fourier series [**3**, p. 153].

Write $f(x)$ in the form $f_1(x)+f_2(x)$ where

$$f_1(x) = \sum_1^N c_{k_\epsilon}(2x)^{1/2}J_\nu(u_kx)$$

and $S_N(x)$ in the form

$$S_N(x) = S_{N_1}(x) + S_{N_2}(x)$$

where $S_{N_1}(x)$ and $S_{N_2}(x)$ are the Nth partial sums of the Fourier-Bessel series of $f_1(x)$ and $f_2(x)$ respectively. Then

$$\int_0^1 |f(x) - S_N(x)|^p dx$$

$$= \int_0^1 |f_1(x) - S_{N_1}(x) + f_2(x) - S_{N_2}(x)|^p dx$$

$$\leqq \int_0^1 |f_1(x) - S_{N_1}(x)|^p dx + \int_0^1 |f_2(x) - S_{N_2}(x)|^p dx;$$

but, if N is large enough, $S_{N_1}(x) = f_1(x)$ since $f_1(x)$ is merely a finite linear combination of functions from the set $\{(2x)^{1/2}J_\nu(u_k x)\}$. Finally

$$\int_0^1 |f(x) - S_N(x)|^p dx \leqq \int_0^1 |f_2(x)|^p dx + \int_0^1 |S_{N_2}(x)|^p dx$$

$$\leqq \epsilon + A \int_0^1 |f_2(x)|^p dx = (A + 1)\epsilon.$$

References

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3. A. Zygmund, *Trigonometrical series*, Warsaw, 1935.

4. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge, 1944.

Cornell University