NOTES ON A PAPER BY SANOV

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Introduction. In this paper a proof is given of some results obtained by Sanov (see [2]). Some identities are used which are of interest in themselves.

Let $F=\{u, v\}$ be a free group generated by u, v. Let

$$(x, y) = x^{-1}y^{-1}xy$$

for x, yF. If S, T are subgroups of F, let

$$(S, T) = \{(s, t), s \in S, t \in T\}.$$

Let

$$F(k) = \{x^k, x \in F\}, \qquad F_1 = F, \qquad F_k = (F_{k-1}, F).$$

Let

$$(u, v, 1) = (u, v); \qquad (u, v, n) = ((u, v, n-1), v).$$

Then Sanov [2] proved that

$$(1) \qquad (u, v, \alpha p^\alpha - 1)^{p^{\beta-\alpha}} \in F(p^\beta)F_{\alpha p^\alpha + 1}, \qquad \beta, \alpha = 1, 2, \cdots$$

where p is an arbitrary prime.

In this paper a proof simpler than Sanov's is given for the case $p=2$ and $p=3$, with $\alpha=1$, and β arbitrary. It also gives a result if instead of p being a prime, p is divisible by 2 or 3. Identities (4), (6), (9) and (10) below are of interest in themselves.

THEOREM 1. *Let $F=\{u, v\}$ be a free group generated by u, v. Let $k=2n$. Then*

$$(2) \qquad (u, v)^{k/2} \in F(k)F_3 \qquad k = 2, 4, 6, \cdots$$

and, in particular, if $k=2^\beta$, then

$$(3) \qquad (u, v)^{2^{\beta-1}} \in F(2^\beta)F_3 \qquad \beta = 1, 2, \cdots.$$

Comment. (3) is (1) for $\alpha=1$, $p=2$.

PROOF. By induction on k

$$(4) \qquad \begin{aligned} d &= (u, v)(v, u^2) \cdots (v^{j-1}, u^j)(u^j, v^j) \cdots (u^{k-1}, v^{k-1}) \\ &= (u^{-1}v^{-1})^k(vuv^{-1})^k v^k \in F(k), \qquad k = 2, 3, \cdots \end{aligned}$$

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can be proved.

Let $R=\{x, y\}$ be a free associative ring generated by x, y and let u, v correspond to $1+x$, $1+y$ respectively (see [1] or Theorem M1 in [3]). Then (4) will correspond to

$$1 + P_2(x, y) + \cdots$$

where P_2 is a polynomial in x, y of degree 2. Since (u^i, v^j) corresponds to $1+ij[x, y]+\cdots$ where $[x, y]=xy-yx$,

$$\begin{aligned} P_2(x, y) &= (1-(1\cdot 2-2\cdot 2)+\cdots+(-(j-1)\cdot j+j\cdot j)+\cdots \\ &\qquad\qquad + (k-1)(k-1))[x, y] \\ &= (1+2+\cdots+j+\cdots+(k-1))[x, y] \\ &= ((k-1)k/2)[x, y]. \end{aligned}$$

Hence $d\equiv(u, v)^{k(k-1)/2}$ mod F_3, i.e.,

$$(u, v)^{k(k-1)/2} \in F(k)F_3 \tag{5}$$

since $d\in F(k)$. If k is odd, (5) is a trivial statement. If k is even, then for $k=2n$,

$$k(k-1)/2 = n(2n-1) \equiv -n \equiv n \bmod 2n$$

i.e., for k even, $(u, v)^{k/2}\in F(k)F_3$. q.e.d.

Comment. Theorem 1 can also be proved using the identity

$$\begin{aligned} v^{-k}u^{-k}(uv)^k &= (v, u^{k-1})((v, u^{k-1}), v^{k-1})\cdots \\ &\qquad (v, u_j)((v, u_j), v_j)\cdots(v, u)((v, u), v). \quad k = 2, 3, \cdots. \end{aligned} \tag{6}$$

THEOREM 2. *Let $F=\{u, v\}$ be a free group generated by u, v. Let $k=3n$. Then*

$$((u, v), v)^{k/3} \in F(k)F_4 \tag{7}$$

and, in particular, if $k=3^\beta$, then

$$((u, v), v)^{3^{\beta-1}} \in F(3^\beta)F_4. \tag{8}$$

Comment. Note that (8) is (1) with $\alpha=1$, $p=3$.

PROOF.

$$\begin{aligned} f &= (v^{-1}u^{-1})^k(uvuv^{-1}u^{-1})^k(uvu^{-2})^{-k}u^k \\ &= (v^{-1}u^{-1})^{k-2}v^{-1}u^{k-2}(vu^{-1})^{k-2}vu^{k-2} \qquad (9) \\ &= (v, u)(u, v^2)(v^2, u^2)\cdots(v^i, u^i)(u^i, v^{i+1})\cdots(u^{k-2}, v^{k-1}) \\ &\quad \cdot(v^{k-2}, u)(u, v^{k-3})\cdots(v^{k-j-1}, u^j)(u^j, v^{k-j-2})\cdots(u^{k-3}, v)(v, u^{k-2}) \qquad (10) \\ &\in F(k)\cap F(k-2) \qquad\qquad k = 3, 4, \cdots \end{aligned}$$

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can be proved.

$$(11)\quad \begin{aligned}&(v^{-1}u^{-1})^{k-2}v^{-1}\\ &\quad = (v, u)(u, v^2) \cdots (v^j, u^j)(u^j, v^{j+1}) \cdots (u^{k-2}, v^{k-1})v^{-(k-1)}u^{-(k-2)}\end{aligned}$$

which can be proved by induction is useful in proving (10). (11) is a variation of (4).

$$v^{-(k-1)}(v^{-j}, u^j)(u^j, v^{-j-1})v^{k-1} = (v^{k-j-1}, u^j)(u^j, v^{k-j-2})$$

is also useful.

That $f \in F_2$ is obvious. To show $f \in F_3$, map $F = \{u, v\}$ into the free associative ring, $R = \{x, y\}$ with u, v corresponding to $1+x$ and $1+y$ respectively. Investigate $Q_n(x, y)$ where

$$(12)\qquad 1 + Q_2(x, y) + Q_3(x, y) + \cdots$$

is the element of R onto which f is mapped, and $Q_m(x, y)$ is a polynomial of degree m. A somewhat laborious computation shows that

$$(13)\quad \begin{aligned}&Q_2(x, y) = 0,\\ &Q_3(x, y) = [[x, y], y]\binom{k}{3} \quad \text{where} \quad \binom{k}{3} = \frac{k(k-1)(k-2)}{6}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad k = 3, 4, \cdots.\end{aligned}$$

In computing $Q_n(x, y)$ the following lemma is of use:

LEMMA.

$$(u^m, v^n) \equiv (u, v)^{mn}((u, v), u)^{n\binom{m}{2}}((u, v), v)^{m\binom{n}{2}} \bmod F_4,$$

$$(v^m, u^n) = (u, v)^{-mn}((u, v), u)^{-m\binom{n}{2}}((u, v), v)^{-n\binom{m}{2}} \bmod F_4$$

where

$$\binom{m}{2} = \frac{m(m-1)}{2}.$$

Proof of lemma is by induction and uses the identities

$$(xy, z) = (y, (z, x))(x, z)(y, z),$$
$$(x, yz) = (x, z)(x, y)((x, y), z)$$

and the fact that (see [1])

$$(x^r, (y^s, x^t)) \equiv (x, (y, x))^{rst} \equiv ((x, y), x)^{rst} \bmod F_4,$$
$$((x, y), x) \equiv ((y, x), x)^{-1} \bmod F_4$$

where r, s, t are arbitrary integers.

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In computing $Q_3(x, y)$, the identities

$$\binom{n}{3}+\binom{n}{2}=\binom{n+1}{3} \quad \text{and} \quad \binom{n}{2}+n=\binom{n+1}{2}$$

are also useful. Using (13) and investigating

$$\binom{k}{3} \text{ for } k = 6n, 6n+1, \cdots, 6n+5,$$

one finds that the only nontrivial cases are

$$k = 3n, \qquad \binom{k}{3} \equiv k/3 \bmod k,$$

$$k = 3n+2 \quad \binom{k}{3} \equiv (k-2)/3 \bmod k-2,$$

and this shows that for $k=3n$,

$$f \equiv ((u, v), v)^{k/3} \bmod F_4$$

and hence for $k=3n$

$$((u, v), v)^{k/3} \in F(k)F_4 \qquad \text{q.e.d.}$$

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