ON CLOSED CONVEX SURFACES

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1. **Introduction.** The purpose of this paper is to prove the following theorem. *Let S and $\bar{S}$ be two closed orientable convex surfaces of class C''' imbedded in an euclidian space E^3 of three dimensions, and possessing no parabolic points. Let h be a differentiable homeomorphism of S into $\bar{S}$ such that* (a) $\mathrm{II}=\overline{\mathrm{II}}$, II *and* $\overline{\mathrm{II}}$ *being the second fundamental forms of S and $\bar{S}$ respectively,* (b) *such that the Gaussian curvatures K and $\bar{K}$ of S and $\bar{S}$ are equal at corresponding points X and $\bar{X}$, and* (c) *such that the orientations of S and $\bar{S}$ are preserved. Then h is a rigid motion.*

Incidental to the proof of the theorem, we present a simple proof of Liebmann's theorem on the rigidity of the sphere. In seeking an integral formula furnishing a proof of the theorem an integral formula was found which gave a simple proof of the fact that such a surface S, described in the theorem and for which the ratio of the mean to the Gaussian curvature is a constant, is a sphere. Such a surface is of course a "special" Weingarten surface. Chern has proved [**3**] that all convex special W-surfaces are spheres. Hence our statement is but a special case of Chern's theorem.

Since S and $\bar{S}$ are orientable, we may assume that their second fundamental forms are positive definite.

2. **Exterior forms on** S. Let $0-I_1$, I_2, I_3 be a fixed orthogonal frame in E^3. Let (x^1, x^2, x^3) be the coordinates of a point X in E^3 with respect to this orthogonal frame. The vector equation of the surface S has the form

$$X = X(u^1, u^2), \tag{2.1}$$

wherein the components (x^1, x^2, x^3) of the position vector X are of class C''' in a simply connected domain D of a parameter plane. Moreover the vector $X_1 \times X_2$ wherein $X_\alpha = (\partial X/\partial u^\alpha)$, is not a null vector for any point of D.

We shall use the usual summation convention: repeated indices indicating summation over the range of the indices. We shall let the roman letters have the range 1, 2, 3 and the greek letters the range 1, 2.

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Let E_3 be the unit normal vector of S. The first and second fundamental forms of S are given by

$$\begin{aligned} \mathrm{I} &= (dX)^2 = X_\rho \cdot X_\sigma du^\rho du^\sigma = g_{\rho\sigma} du^\rho du^\sigma, \\ \mathrm{II} &= -dE_3 \cdot dX = d_{\rho\sigma} du^\rho du^\sigma. \end{aligned} \tag{2.2}$$

Let $X-E_1, E_2, E_3$ be a frame, to be called *a conjugate frame*, such that $(E_1, E_2, E_3)>0$, and such that if

$$E_1 = U^\rho X_\rho, \qquad E_2 = V^\rho X_\rho, \tag{2.3}$$

then

$$d_{\rho\sigma} U^\rho U^\sigma = 1, \qquad d_{\rho\sigma} U^\rho V^\sigma = 0, \qquad d_{\rho\sigma} V^\rho V^\sigma = 1, \tag{2.4}$$

wherein U^α, V^α are functions of u^1u^2 Class C''. Conditions (2.4) imply that the tangent vectors E_1, E_2 form an orthonormal frame with respect to the metric II. They are conjugate vectors in the sense of Dupin.

Let

$$\overset{2}{E_1} = a, \qquad E_1 \cdot E_2 = b, \qquad \overset{2}{E_2} = c. \tag{2.5}$$

Let ω^1, ω^2 be two forms on S defined by

$$\omega^1 = U_\rho du^\rho, \qquad \omega^2 = V_\rho du^\rho, \qquad U_\alpha = d_{\alpha\rho} U^\rho, \qquad V_\alpha = d_{\alpha\rho} V^\rho. \tag{2.6}$$

From (2.3) and (2.6) we find that

$$X_\alpha = U_\alpha E_1 + V_\alpha E_2, \qquad du^\alpha = U^\alpha \omega^1 + V^\alpha \omega^1. \tag{2.7}$$

Hence

$$dX = X_\rho du^\rho = (U_\rho E_1 + V_\rho E_2)(U^\rho \omega^1 + V^\rho \omega^2) = \omega^i E_i, \ \omega^3 = 0. \tag{2.8}$$

We also write

$$dE_i = \omega_i^{\,j} E_j, \qquad \omega_3^{\,3} = 0. \tag{2.9}$$

Taking exterior differentials of (2.8) and (2.9), and using (2.8) and (2.9) we find that

$$d\omega^i = \omega^j \wedge \omega_j^{\,i}, \qquad d\omega_j^{\,i} = \omega_j^{\,k} \wedge \omega_k^{\,i}. \tag{2.10}$$

Equations (2.10) are of course the conditions of compatibility of (2.8) and (2.9).

Using (2.4) and (2.5) we find that the first and second fundamental forms of S are

$$\mathrm{I} = a(\omega^1)^2 + 2b\omega^1\omega^2 + c(\omega^2)^2, \qquad \mathrm{II} = (\omega^1)^2 + (\omega^2)^2. \tag{2.11}$$

It follows from (2.11) that the mean H of the principal normal curvatures and the Gaussian curvature K of S are given by

$$2H = (a + c)K, \qquad (ac - b^2)K = 1. \tag{2.12}$$

Since $\omega^3 = 0$, we find from the first of (2.10) that

$$\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0.$$

Hence ω_1^3 and ω_2^3 must have the form

$$\omega_1^3 = p\omega^1 + q\omega^2, \qquad \omega_2^3 = q\omega^1 + r\omega^2. \tag{2.13}$$

Since $E_1 \cdot E_3 = 0$, $E_2 \cdot E_3 = 0$, it follows from (2.9) that

$$\omega_3^i E_1 \cdot E_i + \omega_1^i E_i \cdot E_3 = 0, \qquad \omega_3^i E_2 \cdot E_i + \omega_2^i E_i \cdot E_3 = 0.$$

Hence

$$\omega_1^3 = -(a\omega_3^1 + b\omega_3^2), \qquad \omega_2^3 = -(b\omega_3^1 + c\omega_3^2). \tag{2.14}$$

From (2.2) the second fundamental form of S is given by

$$\mathrm{II} = -dE_3 \cdot dX = -\left[(a\omega_3^1 + b\omega_3^2)\omega^1 + (b\omega_3^1 + c\omega_3^2)\omega^2\right] = \omega_1^3\omega^1 + \omega_2^3\omega^2.$$

But from (2.13) and the second of (2.11) we find that $p = r = 1$, $q = 0$. Therefore (2.13) and (2.14) assume the form

$$\omega_1^3 = \omega^1, \quad \omega_2^3 = \omega^2, \quad a\omega_3^1 + b\omega_3^2 = -\omega^1, \quad b\omega_3^1 + c\omega_3^2 = -\omega^2. \tag{2.15}$$

From the last two of (2.15) we find that

$$\omega_3^1 = K(-c\omega^1 + b\omega^2), \qquad \omega_3^2 = K(b\omega^1 - a\omega^2). \tag{2.16}$$

Taking exterior differentials of the first two of (2.15) we find that

$$2\omega_1^1 \wedge \omega^1 = \omega^2 \wedge (\omega_2^1 + \omega_1^2), \quad 2\omega_2^2 \wedge \omega^2 = \omega^1 \wedge (\omega_2^1 + \omega_1^2). \tag{2.17}$$

It follows from (2.17) that ω_1^1, ω_2^2, $\omega_2^1 + \omega_1^2$ have the following form

$$\begin{aligned} \omega_1^1 &= A\omega^1 + B\omega^2, \\ \omega_2^2 &= A'\omega^1 + B'\omega^2, \\ \omega_2^1 + \omega_1^2 &= 2(B\omega^1 + A'\omega^2). \end{aligned} \tag{2.18}$$

From (2.5) and (2.9) we find that

$$da = 2(a\overset{1}{\omega}_1 + b\overset{2}{\omega}_1), \qquad dc = 2(b\overset{1}{\omega}_2 + c\overset{2}{\omega}_2),$$
$$(2.19) \qquad db = b(\overset{1}{\omega}_1 + \overset{2}{\omega}_2) + a\overset{1}{\omega}_2 + c\overset{2}{\omega}_1.$$

Hence

$$d(ac - b^2) = 2(\overset{1}{\omega}_1 + \overset{2}{\omega}_2)(ac - b^2).$$

Therefore

$$(2.20) \qquad dK = -2(\overset{1}{\omega}_1 + \overset{2}{\omega}_2)K.$$

3. **An associated Riemannian space** $\mathfrak{R}$. Consider a Riemannian space $\mathfrak{R}$ defined over D whose metric is given by

$$(3.1) \qquad ds^2 = \mathrm{II} = (\omega^1)^2 + (\omega^2)^2.$$

We shall call this space *the associated Riemannian space.*

As is well known for Riemannian spaces [**2**] there exists an unique form $\psi_1^2(= -\psi_2^1)$ such that

$$(3.2) \qquad d\overset{1}{\omega} = \overset{2}{\omega} \wedge \overset{1}{\psi}_2, \qquad d\overset{2}{\omega} = \overset{1}{\omega} \wedge \overset{2}{\psi}_1.$$

That this form is unique follows from assuming there are two forms ψ_2^1, $\bar{\psi}_2^1$ satisfying (3.2). That is, not only does (3.2) hold but also

$$d\overset{1}{\omega} = \overset{2}{\omega} \wedge \overset{1}{\bar{\psi}}_2, \qquad d\overset{2}{\omega} = \overset{1}{\omega} \wedge \overset{2}{\bar{\psi}}_1.$$

Subtracting (3.2) from the above it follows that

$$\overset{2}{\omega} \wedge (\overset{1}{\bar{\psi}}_2 - \overset{1}{\psi}_2) = 0,$$
$$\overset{1}{\omega} \wedge (\overset{2}{\bar{\psi}}_1 - \overset{2}{\psi}_1) = 0.$$

But since ω^1, ω^2 are linearly independent, $\bar{\psi}_1^2 = \psi_1^2$. From the first of (2.10) with first $i=1$, then $i=2$ and using (2.17) we find readily that

$$(3.3) \qquad \overset{2}{\psi}_1 = \frac{1}{2}(\overset{2}{\omega}_1 - \overset{1}{\omega}_2)$$

satisfies (3.2) and hence is the desired unique form.

Using (2.10) we find that

$$d\overset{2}{\psi}_1 = \frac{1}{2}\left[(\overset{1}{\omega}_1 - \overset{2}{\omega}_2) \wedge (\overset{1}{\omega}_2 + \overset{2}{\omega}_1) + \overset{1}{\omega} \wedge \overset{2}{\omega}_3 - \overset{2}{\omega} \wedge \overset{1}{\omega}_3\right].$$

Hence from (2.12), (2.16) and (2.18) we find that

$$d\psi_1^2 = -(G+H)\omega^1 \wedge \omega^2, \tag{3.4}$$

wherein H is the mean curvature of S and

$$G = A'(A' - A) + B(B - B'). \tag{3.5}$$

It follows that

$$\mathfrak{R} = G + H \tag{3.6}$$

is the Gaussian curvature of the associated Riemannian space. We shall call this curvature *the associated curvature* of S. The associated curvatures of all surfaces having the same second fundamental forms are of course the same.

4. **Integral formulas on** S. Before developing the formulas we shall use to prove the theorem, it will be necessary to consider the effect on the functions a, b, c and the forms ω^i, ω_j^i by a change of conjugate frame.

Let $F' = X - E_1'$, E_2', E_3', $E_3' = E_3$ be a second conjugate frame. Letting

$$E_1' = U'^\rho X_\rho, \qquad E_2' = V'^\rho X_\rho,$$

and noting that E_1', E_2' have the same orientation as E_1, E_2, and are also orthonormal with respect to II (cf. 2.4) we may write

$$E_1' = E_1 \cos\theta + E_2 \sin\theta, \qquad E_2' = -E_1 \sin\theta + E_2 \cos\theta, \tag{4.1}$$

θ being a function of u^1, u^2 of class C'.

The functions a', b', c' corresponding to a, b, c are readily found to be given by

$$\begin{aligned} a' &= a\cos^2\theta + b\sin 2\theta + c\sin^2\theta, \\ 2b' &= (c-a)\sin 2\theta + 2b\cos 2\theta, \\ c' &= a\sin^2\theta - b\sin 2\theta + c\cos^2\theta. \end{aligned} \tag{4.2}$$

Since

$$U'^\alpha = U^\alpha \cos\theta + V^\alpha \sin\theta, \qquad V'^\alpha = -U^\alpha \sin\theta + V^\alpha \cos\theta$$

it follows that, if ϕ^i, ϕ_j^i are the forms corresponding to ω^i, ω_j^i,

$$\phi^1 = \omega^1 \cos\theta + \omega^2 \sin\theta, \qquad \phi^2 = -\omega^1 \sin\theta + \omega^2 \cos\theta. \tag{4.3}$$

Moreover

$$\phi^1 \wedge \phi^2 + \omega^1 \wedge \omega^2. \tag{4.4}$$

Direct computation from (4.1) gives

$$
\begin{aligned}
\overset{1}{\phi}_1 &= \overset{1}{\omega}_1 \cos^2\theta + \frac{1}{2}(\overset{1}{\omega}_2 + \overset{2}{\omega}_1)\sin 2\theta + \overset{2}{\omega}_2 \sin^2\theta,\\
\overset{2}{\phi}_2 &= \overset{1}{\omega}_1 \sin^2\theta - \frac{1}{2}(\overset{1}{\omega}_2 + \overset{2}{\omega}_1)\sin 2\theta + \overset{2}{\omega}_2 \cos^2\theta,\\
\overset{2}{\phi}_1 &= \frac{1}{2}(\overset{2}{\omega}_2 - \overset{1}{\omega}_1)\sin 2\theta + \overset{2}{\omega}_1 \cos^2\theta - \overset{1}{\omega}_2 \sin^2\theta + d\theta, \qquad (4.5)\\
\overset{1}{\phi}_2 &= \frac{1}{2}(\overset{2}{\omega}_2 - \overset{1}{\omega}_1)\sin 2\theta - \overset{2}{\omega}_1 \sin^2\theta + \overset{1}{\omega}_2 \cos^2\theta - d\theta,\\
\overset{1}{\phi}_3 &= \overset{1}{\omega}_3 \cos\theta + \overset{2}{\omega}_3 \sin\theta, \quad \overset{2}{\phi}_3 = -\overset{1}{\omega}_3 \sin\theta + \overset{2}{\omega}_3 \cos\theta.
\end{aligned}
$$

It follows from the third and fourth of (4.5) that $d\psi_1^2$ is independent of the conjugate frame F, a fact which is geometrically evident.

We find it convenient at this point to note, using (2.4) and (2.6), that

$$\omega^1 \wedge \omega^2 = U_\rho V_\sigma du^\rho \wedge du^\sigma = (U_1V_2 - U_2V_1)du^1 \wedge du^2.$$

We note (4) that

$$U_1V_2 - U_2V_1 = d^{1/2},$$

wherein $d = \det(d_{\alpha\beta})$. Hence

$$\omega^1 \wedge \omega^2 = d^{1/2}du^1 \wedge du^2 = K^{1/2}g^{1/2}du^1 \wedge du^2 = K^{1/2}dA,$$

dA being "the element of area" of S.

Let us now define auxiliary functions y_i by the formulas

$$y_i = X \cdot E_i. \qquad (4.6)$$

We find readily that

$$dy_1 = a\overset{1}{\omega} + b\overset{2}{\omega} + \overset{j}{\omega}_1 y_j, \qquad dy_2 = b\overset{1}{\omega} + c\overset{2}{\omega} + \overset{j}{\omega}_2 y_j. \qquad (4.7)$$

We note from (4.2) and (4.5) that the following forms are independent of the frame F:

$$
\begin{aligned}
\omega_1 &= K^{1/2}(y_1\overset{2}{\omega} - y_2\overset{1}{\omega}),\\
\omega_2 &= K^{1/2}[y_1(b\overset{1}{\omega} + c\overset{2}{\omega}) - y_2(a\overset{1}{\omega} + b\overset{2}{\omega})], \qquad (4.8)\\
\omega_3 &= K^{1/2}(z_1\overset{2}{\omega} - z_2\overset{1}{\omega}),\\
\omega_4 &= K^{1/2}[2b(\overset{1}{\omega}_1 - \overset{2}{\omega}_2) + (c-a)(\overset{1}{\omega}_2 + \overset{2}{\omega}_1)],
\end{aligned}
$$

wherein

$$z_1 = (a - c)y_1 + 2by_2, \qquad z_2 = 2by_1 + (c - a)y_2.$$

Stokes' formula applied to a linear form ω may be written as

$$\int_C \omega = \iint_R d\omega,$$

wherein C is the boundary of the simply connected region R. Applying this formula to the forms (4.8) in turn, and recalling that S was assumed closed and convex, we find that

$$\begin{aligned} &\iint_S (H + Ky_3)dA = 0, \qquad \iint_S (1 + Hy_3)dA = 0, \\ &\iint_S K(p + 2f^\rho y_\rho)dA = 0, \quad \iint_S K[Kp + 4(a + c)G]dA = 0, \end{aligned} \tag{4.9}$$

wherein

$$\begin{aligned} p &= a^2 + c^2 - 2ac + 4b^2, \\ f^1 = aA + 2bB + cA', &\qquad f^2 = aB + 2bA' + cB'. \end{aligned} \tag{4.10}$$

The first and second of (4.9) are of course the familiar formulas associated with closed surfaces, the first being Minkowski's formula [**2**].

The first of (4.10) using (2.12) may be written in the form

$$p = (a - c)^2 + 4b^2 = (a + c)^2 - 4(ac - b^2) = 4(H^2 - K)/K^2. \tag{4.11}$$

Hence $p \geqq 0$ on D, the equality holding only at an umbilical point of S.

Using (3.6) we may write the fourth of (4.9) in the form

$$\iint_S H(2\Re - H)dA = \iint_S KdA. \tag{4.12}$$

Formula (4.12) relates the mean and associated curvatures of S with the *curvatura integra*, and hence to the genus of S.

Consider now the first of (2.12) written in the form

$$2H/K = a + c.$$

From (2.18) and (2.19) we find readily that

$$d(a + c) = 2(f^1\omega^1 + f^2\omega^2).$$

Therefore in case the ratio of the mean to the Gaussian curvature is a constant, the third of (4.9) assumes the form

$$\iint_S KpdA = 0.$$

Hence since $p \geqq 0$, it follows that every point of S is an umbilic, and S is a sphere.

5. **The proof of the theorem.** The formulas developed in the previous sections have analogous forms for the surface $\bar{S}$ of the theorem. We shall denote the corresponding expressions for $\bar{S}$ by the same but barred letters.

The homeomorphism $h: S \to \bar{S}$ induces a homeomorphism h^* on contravariant tensors at a point X of S and a homeomorphism h_* on covariant tensors at a point X of S into contravariant and covariant tensors respectively at $\bar{X} = hX$ on $\bar{S}$.

Let $F = X - E_1, E_2, E_3$ be a given conjugate frame of S. We take for the frame of $\bar{S}$, the image of F, that is

$$\bar{F} = hX - h^*E_1, h^*E_2, \bar{E}_3,$$

$\bar{E}_3$ being the unit normal vector of $\bar{S}$. This frame $\bar{F}$ is a conjugate frame on $\bar{S}$ since the form II is preserved under h. Using such corresponding frames on S and $\bar{S}$, it follows that

$$\bar{\omega}^i = h_* \omega^i. \tag{5.1}$$

By assumption $K = \bar{K}$, hence from (2.20)

$$\bar{\omega}_1^{\,1} + \bar{\omega}_2^{\,2} = \omega_1^{\,1} + \omega_2^{\,2}.$$

From (2.18) and its analogue for $\bar{S}$, it follows that

$$\bar{A} + \bar{A}' = A + A', \qquad \bar{B} + \bar{B}' = B + B'. \tag{5.2}$$

Since $\mathrm{II} = \bar{\mathrm{II}}$, from (3.1), (3.3) and (3.5) it follows that

$$\bar{\omega}_1^{\,2} - \bar{\omega}_2^{\,1} = \omega_1^{\,2} - \omega_2^{\,1}. \tag{5.3}$$

Moreover from (4.2) and the analogue of (4.5) for $\bar{S}$, the form

$$\bar{\omega} = K^{1/2}[2b(\bar{\omega}_1^{\,1} - \bar{\omega}_2^{\,2}) + (c - a)(\bar{\omega}_2^{\,1} + \bar{\omega}_1^{\,2})]$$

is independent of the frame F, and hence is a meaningful linear form on S. Using (5.1), (5.2) and (5.3) we find that

$$d\bar{\omega} = K[C'K + 4(a + c)G']dA, \tag{5.4}$$

wherein

$$\begin{aligned} C' &= 2\delta + 4(H\overline{H} - K)/K^2, \\ 2G' &= 2(A'\overline{A}' + B\overline{B}) - \overline{A}A' - A\overline{A}' - B\overline{B}' - \overline{B}B', \\ \delta &= \begin{vmatrix} a - \bar{a} & b - \bar{b} \\ b - \bar{b} & c - \bar{c} \end{vmatrix}. \end{aligned}$$

Application of Stokes' formula to the form $\bar{\omega}$ over the closed surface S gives

$$\iint_S K[C'K + 4(a + c)G']dA = 0. \tag{5.5}$$

Subtracting the last of (4.9) from (5.5), we obtain on using (2.12) and (3.6)

$$\iint_S [K^2\delta + 2H(2G' - G - \overline{G})]dA = 0. \tag{5.6}$$

Defining Δ by the formula

$$\Delta = \begin{vmatrix} A - A' - (\overline{A} - \overline{A}') & B - B' - (\overline{B} - \overline{B}') \\ B - \overline{B} & A' - \overline{A}' \end{vmatrix}, \tag{5.7}$$

we find that

$$\Delta = 2G' - G - \overline{G}.$$

Hence (5.6) assumes the form

$$\iint_S (K^2\delta + 2H\Delta)dA = 0. \tag{5.8}$$

Since $\bar{a}\bar{c} - \bar{b}^2 = ac - b^2$, it is known [1] that $\delta \leqq 0$ over D, the equality holding if and only if $a = \bar{a}$, $b = \bar{b}$, $c = \bar{c}$. Moreover use of (5.2) enables us to write (5.7) in the form

$$\Delta = -2\begin{vmatrix} A' - \overline{A}' & \overline{B} - B \\ B - \overline{B} & A' - \overline{A}' \end{vmatrix}.$$

It follows that $\Delta \leqq 0$. From the fact that $\delta \leqq 0$ over D, and from (5.8) it follows that

$$\iint_S H\Delta dA \geqq 0,$$

and from $\Delta \leqq 0$, that

$$\iint_S H\Delta dA \leqq 0.$$

Hence

$$\iint_S H\Delta dA = 0; \quad \text{therefore} \quad \iint_S K^2\delta dA = 0.$$

Hence $\delta = 0$, and the first fundamental forms of S and $\bar{S}$ are the same. Hence the homeomorphism h is an isometry and therefore a rigid motion, as was to be proved.

Suppose that the Gaussian curvature K is a constant. From (2.20) and (2.18) it follows that

$$A + A' = 0, \qquad B + B' = 0.$$

We may write (3.5) in the form

$$G = 2(A^2 + B^2) \geqq 0.$$

Since $p \geqq 0$, $G \geqq 0$ it follows from the last of (4.9) that

$$G = 0, \qquad p = 0.$$

Hence S is a sphere. This furnishes the promised simple proof of Liebmann's theorem.

We observe that if the associated curvature of S is equal to its mean curvature then $G = 0$, and S is a sphere.

References

1. S. S. Chern, *Topics in differential geometry* (mimeographed notes), Institute for Advanced Study, 1951.

2. ———, *Introduction to differential geometry* (mimeographed notes), the University of Chicago.

3. ———, *Some new characterizations of the euclidian sphere*, Duke Math. J. vol. 12 (1945) pp. 279–290.

4. V. G. Grove, *On congruences and conjugate nets*, Amer. J. Math. vol. 69 (1947) pp. 59–69.

5. S. Nakajima, *Über characterische Eigenschaften der Kugel*, Tôhoku Math. J. vol. 26 (1926) pp. 361–364.

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