COMMUTATIVE SUBDIRECTLY IRREDUCIBLE RINGS

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Introduction. A subdirectly irreducible ring is one in which the intersection of all the nonzero ideals is a nonzero ideal. Such rings are important not only because every ring is isomorphic to a subdirect sum of subdirectly irreducible rings, but also because the theory of rings without chain conditions uses the concept heavily. Our major knowledge of such rings is contained in [1],[1] where Professor McCoy showed that every commutative subdirectly irreducible ring is one of three kinds. We shall classify them as

Type (α). Fields.

Type (β). Every element is a divisor of zero.

Type (γ). There exist both nondivisors of zero and nilpotent elements.

We shall restrict ourselves to the commutative case and henceforth not repeat its assumption. We shall employ the following notation:

A = the commutative subdirectly irreducible ring.

D = the set of all divisors of zero of A.

J = the Jacobson radical of A.

N = the maximal nilideal of A.

N^* = the set of elements that annihilate N.

D^* = the set of elements that annihilate D.

Q = the unique minimal ideal of A that is contained in every nonzero ideal of A.

As in [2] we shall say that A is bound to its maximal nilideal if $N^* \leqq N$.

Rings of type (β) were considered in [2] and it was shown that they are bound to N. In addition if they satisfy either the descending or the ascending chain condition, they are nilpotent.

The present note deals mainly with rings of type (γ). We show that they are all bound to N and this yields the fact that every commutative ring is isomorphic to a subdirect sum of subdirectly irreducible rings some of which are fields, the others being bound to their maximal nilideals and therefore to their Jacobson radical.[2]

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[1] Numbers in square brackets refer to the bibliography at the end of the paper.

[2] This extends, in the commutative case, a result of Andrunakievic (Doklady Akademii Nauk. SSSR. (N.S.) vol. 98 (1954) pp. 329–332) that every ring with the descending chain condition on two-sided ideals, is a direct sum of a finite number of

Furthermore we show that a ring of type (γ) with either the descending or the ascending chain condition has $D=N$, thus extending a result of McCoy [1]; and that every such ring possesses a unity element. This gives us the fact that every commutative subdirectly irreducible ring with either the descending or the ascending chain condition is a field, is nilpotent or has a unity element and is a field modulo its maximal nilideal. Several other weaker conditions are shown to yield the existence of a unity element. Finally, a major part of the paper is devoted to an example of a ring of type (γ) without a unity element.

1. The fundamental theorem on rings of type (γ) is (see [1]):

THEOREM a. *A ring of type* (γ) *is subdirectly irreducible if and only if*:

1. *The set of elements of A that annihilate D is precisely Q, a principal ideal $=(j)$.*
2. *The set of elements of A that annihilate Q is precisely D.*
3. *The quotient ring $A-D$ is a field.*
4. *For every element d in D, which is not in Q, there exists an element d' in D, d' not in Q, such that $dd'=j$.*

Using a technique of [2] we first prove:

THEOREM 1. *Every subdirectly irreducible ring A of type* (γ) *is bound to its maximal nilideal N.*

PROOF. Let x be any element in N^*. That is, $xN=0$. Then clearly, x is in D. If x is not in N, x^2 is not in N, and since x is in D, x^2 is in D. By Theorem a, number 4, there exists an element x'' in D such that $x^2\cdot x''=j$. Consider the element $x\cdot x''$. It is in N since $(x\cdot x'')^2=x^2\cdot x''\cdot x''=j\cdot x''=0$, since j annihilates D. However $x\cdot xx''=j\neq 0$ which contradicts $xN=0$. Therefore x is in N, $N^*\leqq N$, A is bound to N.

Since every subdirectly irreducible ring of type (β) is bound to N, [2], we have

COROLLARY. *Every subdirectly irreducible ring is either a field or is bound to its maximal nilideal.*

We now turn our attention to the existence of a unity element and the relationship between D and N.

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simple rings with unit and a bound ring; as well as the result of Brown and McCoy (Proc. Amer. Math. Soc. vol. 1 (1950) pp. 165–171) that a ring with the descending chain condition on right ideals is a direct sum of a semisimple and a bound ring. This was done for finite dimensional algebras by Hall (Trans. Amer. Math. Soc. vol. 48 (1940) pp. 391–404).

LEMMA 1. *For every subdirectly irreducible ring A, A has a unity element if and only if $A-N$ has a nonzero idempotent.*

PROOF. One way is trivial. Conversely if $A-N$ has a nonzero idempotent then A has a nonzero idempotent (see the more general result as part of Hall, Ann. of Math. vol. 40 (1939) p. 368, Theorem 6.1). Let S be a nonzero ideal of A which has this idempotent as a unity element. Then $A=S\oplus S^*$, where S^* is the annihilator of S. Since A is subdirectly irreducible and $S\neq 0$, $S^*=0$ and therefore $A=S$, A has a unity element.

COROLLARY. *If a subdirectly irreducible ring A of type (γ) has $D=N$ then A has a unity element.*

PROOF. Since $A-D$ is a field (Theorem a, number 3), $A-D$ has a nonzero idempotent. Then if $N=D$, $A-N$ has one, and therefore by the lemma, A has a unity element.

From [1] we learn that if A is a subdirectly irreducible ring of type (γ) and if A has the descending chain condition, then $D=N$. By the above corollary, A has a unity element. We now extend this to rings with the ascending chain condition.

THEOREM 2. *If A is a subdirectly irreducible ring of type (γ) and if A has the ascending chain condition, then $D=N$.*

PROOF. If $D\neq N$, take x in D, x not in N. Then all the powers $x^2, x^3, \cdots, x^n, \cdots$ are in D and not in N, and for every power x^n there exists an element y_n such that $xy_1=x^2y_2=\cdots=x^ny_n=\cdots=j$. Define

$$V_i=\{z: zx^i=0\}.$$

Clearly $V_1<V_2<V_3<\cdots<V_n<\cdots$. Also, since all the elements of V_i are divisors of zero, $V_i\leqq D$, for every i. Since $x^i\cdot D\neq 0$ for every i (since x is not nilpotent), all the V_i are properly contained in D. Now y_1 is not in V_1, but $x\cdot xy_1=xj=0$, and therefore y_1 is in V_2. Similarly y_n is not in V_n, but $x\cdot x^ny_n=xj=0$, and therefore y_n is in V_{n+1}. Therefore the V_i form a properly ascending chain which contradicts the ascending chain condition. Therefore no such x exists, and $D=N$. Therefore we have

THEOREM 3. *If A is a subdirectly irreducible ring of type (γ) and if A has either the descending or the ascending chain condition, then A has a unity element.*

2. **Examples.** Since every subdirectly irreducible ring of type (β) is bound to N and is nilpotent if it has either the descending or the

ascending chain condition we shall first give an example of such a ring which is not nilpotent. It is based on an example in [1].

Let A be an algebra over a field of characteristic 2, with the following basis: $x, x^2, \cdots, x^n, \cdots; x^{t-1}, x^{t-2}, \cdots, x^{t-m}, \cdots$, where both x and t are indeterminates and $x^t=0$. The algebra A is the set of all finite linear combinations. Then the unique minimal ideal $Q=\{x^{t-1}\}$. Since $x^{t-1}\cdot(\sum\alpha_i\cdot x^i+\sum\beta_j\cdot x^{t-j})=0$, every element is a divisor of zero. Thus A is of type (β). To see that x^{t-1} is in fact contained in every ideal, let z be any element of A, $z=\sum_1^n \alpha_i x^i+\sum_1^m \beta_j x^{t-j}$. If the α_i are not all zero, let α_s be the first nonzero α_i (and therefore $=1$) and consider $z\cdot x^{t-s-1}$. Since $x^{t-p}\cdot x^{t-q}=x^t\cdot x^{t-q-p}=0$, $z\cdot x^{t-s-1}=x^s\cdot x^{t-s-1}+\alpha_{s+1}x^{s+1}\cdot x^{t-s-1}+\cdots+\alpha_n x^n x^{t-s-1}=x^{t-1}$. If on the other hand all the α_i are zero, $z=\sum_1^m \beta_j x^{t-j}$. Then if β_m is the last nonzero coefficient (and therefore $=1$) consider $z\cdot x^{m-1}=\beta_1 x^{t-1}x^{m-1}+\cdots+x^{t-m}x^{m-1}=x^{t-1}$. Of course if $m=1$, $z=x^{t-1}$. Therefore x^{t-1} is in every nonzero ideal, and A is subdirectly irreducible. Finally we note that x is not nilpotent and therefore A is not nilpotent.

Turning back now to subdirectly irreducible rings of type (γ) we give an example of such a ring without a unity element.

Let A be an algebra over a field F, characteristic 0, with the following basis: $e, e^2, \cdots, e^n, \cdots, z_0, z_1, z_2, \cdots, z_m, \cdots$, where e is an indeterminate, and A is the set of all finite linear combinations of the e^n's and z_m's with coefficients in F. The multiplication table is:

$$z_i\cdot z_j = 0, \qquad \text{for every } i \text{ and } j;$$

$$e\cdot z_0 = z_0;$$

$$e\cdot z_m = z_m + z_{m-1}, \qquad \text{for } m > 0.$$

We define Q to be the ideal generated by z_0 and will show later that Q is the unique minimal ideal contained in every nonzero ideal. The maximal nilideal N is clearly the ideal generated by $z_0, z_1, \cdots, z_m, \cdots$, and $N^2=0$.

We define D to be the set of all $\sum_{i=1}^n \alpha_i e^i+\sum_{j=0}^m \beta_j z_j$, where $\sum_{i=1}^n \alpha_i=0$. We will show later that D is the set of all divisors of zero.

Clearly A contains nonzero nilpotent elements. It also has nondivisors of zero, namely e. For a simple computation shows that if $e\cdot a=0$, where a is any element in A, then a must be zero. Consequently A is of type (γ).

To see that A is subdirectly irreducible we shall first establish the four parts of Theorem a and then prove that D is the set of all divisors of zero.

(1) $\{x: xD=0\}=Q$.

PROOF. Since $z_0 \cdot z_i=0$ for every i, and since

$$z_0 \cdot \sum_{i=1}^n \alpha_i e^i = z_0 \cdot \sum_{i=1}^n \alpha_i = 0 \text{ if } \sum_{i=1}^n \alpha_i = 0,$$

it is clear that $Q \leqq \{x: xD=0\}$. To show that $\{x: xD=0\} \leqq Q$, we must first establish the formula

$$z_m \cdot (e^2 - e)^m = z_0. \tag{1}$$

We show this by induction. For $m=1$, $z_1 \cdot (e^2-e)^1 = z_1 \cdot e^2 - z_1 \cdot e = z_1 + 2z_0 - z_1 - z_0 = z_0$. We assume that $z_{m-1} \cdot (e^2-e)^{m-1} = z_0$ and we consider $z_m \cdot (e^2-e)^m$. Since $z_m \cdot e = z_m + z_{m-1}$, $z_m \cdot (e-1) = z_{m-1}$, when we multiply by e this becomes $z_m \cdot (e^2-e) = z_{m-1} \cdot e$. Therefore $z_m \cdot (e^2-e)^m = z_m \cdot (e^2-e) \cdot (e^2-e)^{m-1} = z_{m-1} \cdot e \cdot (e^2-e)^{m-1} = e \cdot z_0 = z_0$. This establishes (1). As a consequence of (1) we have

$$z_m \cdot (e^2 - e)^n = 0, \qquad \text{for } n > m. \tag{2}$$

This is clear since $z_m \cdot (e^2-e)^n = z_m \cdot (e^2-e)^m \cdot (e^2-e)^{n-m} = z_0 \cdot (e^2-e)^{n-m} = 0$, because e^2-e is in D.

Now let x be any element such that $xD=0$; $x = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$. Since $(e^2 - e)^k = e^k \cdot (e - 1)^k$ has the sum of its coefficients $= 1 - {}_kC_1 + {}_kC_2 \cdots \pm 1 = (1-1)^k = 0$, $(e^2-e)^k$ is in D for every k. Take $k=m+1$ and consider $x \cdot (e^2-e)^{m+1} = 0$. Since $\sum_0^m \beta_j z_j \cdot (e^2-e)^{m+1} = 0$ by (2), we have $\sum_1^n \alpha_i e^i \cdot (e^2-e)^{m+1} = 0$. This is possible only if all the α_i are zero, and therefore $x = \sum_0^m \beta_j z_j$. If there is a $\beta_m \neq 0$, with $m>0$, consider $0 = x \cdot (e^2-e)^m = \sum_0^m \beta_j z_j \cdot (e^2-e)^m = \beta_m z_0$, by (2) and (1). Since this is impossible, all the β_m's, $m>0$, must be zero, and $x = \beta_0 z_0$, x is in Q.

(2) $\{x: xQ=0\}=D$.

PROOF. As in the proof of (1) it is clear that $D \cdot Q = 0$. To show that $\{x: xQ=0\} \leqq D$, let x be any element such that $xQ=0$, $x = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$. Since $xQ=0$, $xz_0=0$ and therefore $\sum_1^n \alpha_i e^i \cdot z_0 = z_0 \cdot \sum_1^n \alpha_i = 0$. Thus $\sum_1^n \alpha_i = 0$, and x is in D.

(3) $A-D$ is a field.

PROOF. Let x be any element in A, $x = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$. Then

$$x = \sum_1^n \alpha_i \cdot e + \left(-\sum_2^n \alpha_i \cdot e + \sum_2^n \alpha_i \cdot e^i\right) + \sum_0^m \beta_j z_j$$

$$= \alpha \cdot e + g(e) + \sum_0^m \beta_j z_j,$$

where $\alpha = \sum_1^n \alpha_i$ and where $g(e) + \sum_0^m \beta_j z_j$ is in D. Thus $x = \alpha e + d$, with d in D. Therefore $A-D$ is simply the set of all $\alpha e + D$, α in F.

Since $e-e^2$ is in D, $\alpha e\cdot e=\alpha\cdot e^2\equiv\alpha\cdot e$ (modulo D). Therefore $e+D$ is the unity element of $A-D$. Also $(\alpha e+D)(\alpha^{-1}\cdot e+D)=e^2+D=e+D$ and therefore $A-D$ is a field.

(4) If d is in D, not in Q, there exists an element d' in D, not in Q, such that $dd'=z_0$.

PROOF. Let x be any element in D, x not in Q. Then $x=\sum_1^n\alpha_i e^i+\sum_0^m\beta_j z_j$, with $\sum_1^n\alpha_i=0$. If all the α_i's are zero, $x=\sum_0^m\beta_j z_j$, with $\beta_m\neq 0$, $m>0$. Let $x'=\beta_m^{-1}\cdot(e^2-e)^m$. Then $x\cdot x'=\sum_0^m\beta_j z_j\cdot\beta_m^{-1}\cdot(e^2-e)^m=\beta_m z_m\beta_m^{-1}\cdot(e^2-e)^m$, by (2) and this is $=z_0$, by (1). Clearly x' is in D and not in Q.

Therefore we assume that not all of the α_i's are zero, $\alpha_n\neq 0$. Consider the following set of elements of F:

$$\gamma_0=\sum_1^n\alpha_i,\qquad \gamma_1=\sum_1^n {}_iC_1\cdot\alpha_i,\qquad \gamma_2=\sum_2^n {}_iC_2\cdot\alpha_i,\ \cdots$$

$$\gamma_s=\sum_s^n {}_iC_s\cdot\alpha_i,\ \cdots,\ \gamma_n=\sum_n^n {}_iC_n\cdot\alpha_i=\alpha_n.$$

Thus not all of the γ_s are zero and there exists an integer r such that $\gamma_0=\gamma_1=\cdots=\gamma_{r-1}=0$, $\gamma_r\neq 0$. Let $x'=\gamma_r^{-1}\cdot z_r$ and compute

$$\gamma_r\cdot x\cdot x'=x\cdot z_r=\sum_1^n\alpha_i e^i\cdot z_r+\sum_0^m\beta_j z_j\cdot z_r=\sum_1^n\alpha_i e^i\cdot z_r$$
$$=\sum_{i=1}^n\sum_{j=0}^r\alpha_i\cdot {}_iC_j\cdot z_{r-j}$$

where ${}_iC_0$ is taken to be 1 and where ${}_iC_j$, with $i<j$, is taken to be 0. Then

$$\gamma_r\cdot x\cdot x'=z_r\cdot\gamma_0+z_{r-1}\cdot\gamma_1+\cdots+z_1\gamma_{r-1}+z_0\gamma_r=z_0\gamma_r.$$

Therefore $x\cdot x'=z_0$. This establishes (4).

To see that D is the set of all divisors of zero it is sufficient to show that every divisor of zero is in D, since we already know that $D\cdot z_0=0$. Thus let x be any divisor of zero. Then for some y, $xy=0$. If x is not in D, then y must be in D, because $A-D$ is a field. If y is in Q, $y=\alpha z_0$, then $x\cdot z_0=0$ and $xQ=0$. By (2), x is in D. If y is not in Q, then by (4), there exists an element y' in D such that $yy'=z_0$. Then $xyy'=xz_0=0$, $xQ=0$, x is in D by (2).

Therefore by Theorem a, A is a subdirectly irreducible ring.

We must finally prove that A does not possess a unity element. If f is a unity element then it must be congruent to e modulo D, $f=e+d$. Since e is not a unity element ($e\cdot z_1=z_1+z_0\neq z_1$), $d\neq 0$. Then there exists an element d' such that $dd' = z_0$ and therefore $fd' = ed' + z_0 = d'$. That is, $ed' - d' = -z_0$. Now $d = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$, with $\sum_1^n \alpha_i = 0$. If all the α_i are zero, then d' can be taken $= -\beta_m^{-1} \cdot (e^2 - e)^m$, (see proof of (4)). Consequently $e \cdot \beta_m^{-1} \cdot (e^2 - e)^m - \beta_m^{-1} \cdot (e^2 - e)^m = -z_0$, $(e^2 - e)^m \cdot (e - 1) = -\beta_m z_0$. Multiplying by $e \cdot z_{m+1}$ this becomes $(e^2 - e)^{m+1} \cdot z_{m+1} = z_0 = -\beta_m z_0 z_{m+1} \cdot e = 0$, which is impossible.

On the other hand if the α_i are not all zero then d' can be taken $= \gamma_r^{-1} \cdot z_r$, (see proof of (4)), and we have $e \cdot \gamma_r^{-1} \cdot z_r - \gamma_r^{-1} \cdot z_r = -z_0$. Therefore $z_r - ez_r = \gamma_r z_0 = z_r - z_r - z_{r-1} = -z_{r-1}$. This is false unless $r = 1$, $d' = z_1$, $f = e + e - e^2 + \sum_0^m \beta_j z_j$. Then $f \cdot z_2 = 2e \cdot z_2 - e^2 z_2 = 2z_2 + 2z_1 - z_2 - 2z_1 - z_0 = z_2 - z_0 \neq z_2$. Therefore A has no unity element.

We might point out that for subdirectly irreducible rings of type (γ), D is a maximal regular ideal and therefore contains J, the Jacobson radical. In this particular example, $J = N$, since $A - N$ is the set of all finite sums $\sum_1^n \alpha_i e^i$ and this set has zero Jacobson radical.

Since rings of type (β) and (γ) seem to have such similar properties, one might expect a more intimate relationship between them. However the N of a ring of type (γ) is not necessarily subdirectly irreducible, as the main example above proves.

One can however show that every ring of type (β) is contained in a ring of type (γ). The following theorem was pointed out to the author by Professor McCoy. It can be proved for the noncommutative case, and shows further that a ring of type (γ) without a unity element is contained in a ring of type (γ) with a unity element.

THEOREM 4. *Every subdirectly irreducible ring without a unity element, can be embedded in a subdirectly irreducible ring with a unity element.*

As Professor McCoy pointed out, the proof of this theorem follows almost immediately from Lemma 5 in the paper, *On the theory of radicals in a ring*, by M. Nagata, Journal of the Mathematical Society of Japan vol. 3 (1951) pp. 330–344.

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