

# COMMUTATIVE SUBDIRECTLY IRREDUCIBLE RINGS

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**Introduction.** A subdirectly irreducible ring is one in which the intersection of all the nonzero ideals is a nonzero ideal. Such rings are important not only because every ring is isomorphic to a subdirect sum of subdirectly irreducible rings, but also because the theory of rings without chain conditions uses the concept heavily. Our major knowledge of such rings is contained in [1],<sup>1</sup> where Professor McCoy showed that every commutative subdirectly irreducible ring is one of three kinds. We shall classify them as

Type ( $\alpha$ ). Fields.

Type ( $\beta$ ). Every element is a divisor of zero.

Type ( $\gamma$ ). There exist both nondivisors of zero and nilpotent elements.

We shall restrict ourselves to the commutative case and henceforth not repeat its assumption. We shall employ the following notation:

$A$  = the commutative subdirectly irreducible ring.

$D$  = the set of all divisors of zero of  $A$ .

$J$  = the Jacobson radical of  $A$ .

$N$  = the maximal nilideal of  $A$ .

$N^*$  = the set of elements that annihilate  $N$ .

$D^*$  = the set of elements that annihilate  $D$ .

$Q$  = the unique minimal ideal of  $A$  that is contained in every nonzero ideal of  $A$ .

As in [2] we shall say that  $A$  is bound to its maximal nilideal if  $N^* \leq N$ .

Rings of type ( $\beta$ ) were considered in [2] and it was shown that they are bound to  $N$ . In addition if they satisfy either the descending or the ascending chain condition, they are nilpotent.

The present note deals mainly with rings of type ( $\gamma$ ). We show that they are all bound to  $N$  and this yields the fact that every commutative ring is isomorphic to a subdirect sum of subdirectly irreducible rings some of which are fields, the others being bound to their maximal nilideals and therefore to their Jacobson radical.<sup>2</sup>

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<sup>1</sup> Numbers in square brackets refer to the bibliography at the end of the paper.

<sup>2</sup> This extends, in the commutative case, a result of Andrunakievic (Doklady Akademii Nauk. SSSR. (N.S.) vol. 98 (1954) pp. 329-332) that every ring with the descending chain condition on two-sided ideals, is a direct sum of a finite number of

Furthermore we show that a ring of type  $(\gamma)$  with either the descending or the ascending chain condition has  $D = N$ , thus extending a result of McCoy [1]; and that every such ring possesses a unity element. This gives us the fact that every commutative subdirectly irreducible ring with either the descending or the ascending chain condition is a field, is nilpotent or has a unity element and is a field modulo its maximal nilideal. Several other weaker conditions are shown to yield the existence of a unity element. Finally, a major part of the paper is devoted to an example of a ring of type  $(\gamma)$  without a unity element.

1. The fundamental theorem on rings of type  $(\gamma)$  is (see [1]):

**THEOREM a.** *A ring of type  $(\gamma)$  is subdirectly irreducible if and only if:*

1. *The set of elements of  $A$  that annihilate  $D$  is precisely  $Q$ , a principal ideal  $= (j)$ .*
2. *The set of elements of  $A$  that annihilate  $Q$  is precisely  $D$ .*
3. *The quotient ring  $A - D$  is a field.*
4. *For every element  $d$  in  $D$ , which is not in  $Q$ , there exists an element  $d'$  in  $D$ ,  $d'$  not in  $Q$ , such that  $dd' = j$ .*

Using a technique of [2] we first prove:

**THEOREM 1.** *Every subdirectly irreducible ring  $A$  of type  $(\gamma)$  is bound to its maximal nilideal  $N$ .*

**PROOF.** Let  $x$  be any element in  $N^*$ . That is,  $xN = 0$ . Then clearly,  $x$  is in  $D$ . If  $x$  is not in  $N$ ,  $x^2$  is not in  $N$ , and since  $x$  is in  $D$ ,  $x^2$  is in  $D$ . By Theorem a, number 4, there exists an element  $x''$  in  $D$  such that  $x^2 \cdot x'' = j$ . Consider the element  $x \cdot x''$ . It is in  $N$  since  $(x \cdot x'')^2 = x^2 \cdot x'' \cdot x'' = j \cdot x'' = 0$ , since  $j$  annihilates  $D$ . However  $x \cdot xx'' = j \neq 0$  which contradicts  $xN = 0$ . Therefore  $x$  is in  $N$ ,  $N^* \leq N$ ,  $A$  is bound to  $N$ .

Since every subdirectly irreducible ring of type  $(\beta)$  is bound to  $N$ , [2], we have

**COROLLARY.** *Every subdirectly irreducible ring is either a field or is bound to its maximal nilideal.*

We now turn our attention to the existence of a unity element and the relationship between  $D$  and  $N$ .

simple rings with unit and a bound ring; as well as the result of Brown and McCoy (Proc. Amer. Math. Soc. vol. 1 (1950) pp. 165-171) that a ring with the descending chain condition on right ideals is a direct sum of a semisimple and a bound ring. This was done for finite dimensional algebras by Hall (Trans. Amer. Math. Soc. vol. 48 (1940) pp. 391-404).

LEMMA 1. *For every subdirectly irreducible ring  $A$ ,  $A$  has a unity element if and only if  $A - N$  has a nonzero idempotent.*

PROOF. One way is trivial. Conversely if  $A - N$  has a nonzero idempotent then  $A$  has a nonzero idempotent (see the more general result as part of Hall, Ann. of Math. vol. 40 (1939) p. 368, Theorem 6.1). Let  $S$  be a nonzero ideal of  $A$  which has this idempotent as a unity element. Then  $A = S \oplus S^*$ , where  $S^*$  is the annihilator of  $S$ . Since  $A$  is subdirectly irreducible and  $S \neq 0$ ,  $S^* = 0$  and therefore  $A = S$ ,  $A$  has a unity element.

COROLLARY. *If a subdirectly irreducible ring  $A$  of type  $(\gamma)$  has  $D = N$  then  $A$  has a unity element.*

PROOF. Since  $A - D$  is a field (Theorem a, number 3),  $A - D$  has a nonzero idempotent. Then if  $N = D$ ,  $A - N$  has one, and therefore by the lemma,  $A$  has a unity element.

From [1] we learn that if  $A$  is a subdirectly irreducible ring of type  $(\gamma)$  and if  $A$  has the descending chain condition, then  $D = N$ . By the above corollary,  $A$  has a unity element. We now extend this to rings with the ascending chain condition.

THEOREM 2. *If  $A$  is a subdirectly irreducible ring of type  $(\gamma)$  and if  $A$  has the ascending chain condition, then  $D = N$ .*

PROOF. If  $D \neq N$ , take  $x$  in  $D$ ,  $x$  not in  $N$ . Then all the powers  $x^2, x^3, \dots, x^n, \dots$  are in  $D$  and not in  $N$ , and for every power  $x^n$  there exists an element  $y_n$  such that  $xy_1 = x^2y_2 = \dots = x^ny_n = \dots = j$ . Define

$$V_i = \{z: zx^i = 0\}.$$

Clearly  $V_1 < V_2 < V_3 < \dots < V_n < \dots$ . Also, since all the elements of  $V_i$  are divisors of zero,  $V_i \leq D$ , for every  $i$ . Since  $x^i \cdot D \neq 0$  for every  $i$  (since  $x$  is not nilpotent), all the  $V_i$  are properly contained in  $D$ . Now  $y_1$  is not in  $V_1$ , but  $x \cdot xy_1 = xj = 0$ , and therefore  $y_1$  is in  $V_2$ . Similarly  $y_n$  is not in  $V_n$ , but  $x \cdot x^ny_n = xj = 0$ , and therefore  $y_n$  is in  $V_{n+1}$ . Therefore the  $V_i$  form a properly ascending chain which contradicts the ascending chain condition. Therefore no such  $x$  exists, and  $D = N$ . Therefore we have

THEOREM 3. *If  $A$  is a subdirectly irreducible ring of type  $(\gamma)$  and if  $A$  has either the descending or the ascending chain condition, then  $A$  has a unity element.*

2. **Examples.** Since every subdirectly irreducible ring of type  $(\beta)$  is bound to  $N$  and is nilpotent if it has either the descending or the

ascending chain condition we shall first give an example of such a ring which is not nilpotent. It is based on an example in [1].

Let  $A$  be an algebra over a field of characteristic 2, with the following basis:  $x, x^2, \dots, x^n, \dots; x^{t-1}, x^{t-2}, \dots, x^{t-m}, \dots$ , where both  $x$  and  $t$  are indeterminates and  $x^t=0$ . The algebra  $A$  is the set of all finite linear combinations. Then the unique minimal ideal  $Q = \{x^{t-1}\}$ . Since  $x^{t-1} \cdot (\sum \alpha_i \cdot x^i + \sum \beta_j \cdot x^{t-j}) = 0$ , every element is a divisor of zero. Thus  $A$  is of type  $(\beta)$ . To see that  $x^{t-1}$  is in fact contained in every ideal, let  $z$  be any element of  $A$ ,  $z = \sum_1^n \alpha_i x^i + \sum_1^m \beta_j x^{t-j}$ . If the  $\alpha_i$  are not all zero, let  $\alpha_s$  be the first nonzero  $\alpha_i$  (and therefore = 1) and consider  $z \cdot x^{t-s-1}$ . Since  $x^{t-p} \cdot x^{t-q} = x^t \cdot x^{t-p-q} = 0$ ,  $z \cdot x^{t-s-1} = x^s \cdot x^{t-s-1} + \alpha_{s+1} x^{s+1} \cdot x^{t-s-1} + \dots + \alpha_n x^n x^{t-s-1} = x^{t-1}$ . If on the other hand all the  $\alpha_i$  are zero,  $z = \sum_1^m \beta_j x^{t-j}$ . Then if  $\beta_m$  is the last nonzero coefficient (and therefore = 1) consider  $z \cdot x^{m-1} = \beta_1 x^{t-1} x^{m-1} + \dots + x^{t-m} x^{m-1} = x^{t-1}$ . Of course if  $m=1$ ,  $z = x^{t-1}$ . Therefore  $x^{t-1}$  is in every nonzero ideal, and  $A$  is subdirectly irreducible. Finally we note that  $x$  is not nilpotent and therefore  $A$  is not nilpotent.

Turning back now to subdirectly irreducible rings of type  $(\gamma)$  we give an example of such a ring without a unity element.

Let  $A$  be an algebra over a field  $F$ , characteristic 0, with the following basis:  $e, e^2, \dots, e^n, \dots, z_0, z_1, z_2, \dots, z_m, \dots$ , where  $e$  is an indeterminate, and  $A$  is the set of all finite linear combinations of the  $e^n$ 's and  $z_m$ 's with coefficients in  $F$ . The multiplication table is:

$$\begin{aligned} z_i \cdot z_j &= 0, && \text{for every } i \text{ and } j; \\ e \cdot z_0 &= z_0; \\ e \cdot z_m &= z_m + z_{m-1}, && \text{for } m > 0. \end{aligned}$$

We define  $Q$  to be the ideal generated by  $z_0$  and will show later that  $Q$  is the unique minimal ideal contained in every nonzero ideal. The maximal nilideal  $N$  is clearly the ideal generated by  $z_0, z_1, \dots, z_m, \dots$ , and  $N^2=0$ .

We define  $D$  to be the set of all  $\sum_{i=1}^n \alpha_i e^i + \sum_{j=0}^m \beta_j z_j$ , where  $\sum_{i=1}^n \alpha_i = 0$ . We will show later that  $D$  is the set of all divisors of zero.

Clearly  $A$  contains nonzero nilpotent elements. It also has non-divisors of zero, namely  $e$ . For a simple computation shows that if  $e \cdot a = 0$ , where  $a$  is any element in  $A$ , then  $a$  must be zero. Consequently  $A$  is of type  $(\gamma)$ .

To see that  $A$  is subdirectly irreducible we shall first establish the four parts of Theorem a and then prove that  $D$  is the set of all divisors of zero.

$$(1) \{x: xD=0\} = Q.$$

PROOF. Since  $z_0 \cdot z_i = 0$  for every  $i$ , and since

$$z_0 \cdot \sum_{i=1}^n \alpha_i e^i = z_0 \cdot \sum_{i=1}^n \alpha_i = 0 \text{ if } \sum_{i=1}^n \alpha_i = 0,$$

it is clear that  $Q \subseteq \{x: xD=0\}$ . To show that  $\{x: xD=0\} \subseteq Q$ , we must first establish the formula

$$(1) \quad z_m \cdot (e^2 - e)^m = z_0.$$

We show this by induction. For  $m=1$ ,  $z_1 \cdot (e^2 - e)^1 = z_1 \cdot e^2 - z_1 \cdot e = z_1 + 2z_0 - z_1 - z_0 = z_0$ . We assume that  $z_{m-1} \cdot (e^2 - e)^{m-1} = z_0$  and we consider  $z_m \cdot (e^2 - e)^m$ . Since  $z_m \cdot e = z_m + z_{m-1}$ ,  $z_m \cdot (e - 1) = z_{m-1}$ , when we multiply by  $e$  this becomes  $z_m \cdot (e^2 - e) = z_{m-1} \cdot e$ . Therefore  $z_m \cdot (e^2 - e)^m = z_m \cdot (e^2 - e) \cdot (e^2 - e)^{m-1} = z_{m-1} \cdot e \cdot (e^2 - e)^{m-1} = e \cdot z_0 = z_0$ . This establishes (1). As a consequence of (1) we have

$$(2) \quad z_m \cdot (e^2 - e)^n = 0, \quad \text{for } n > m.$$

This is clear since  $z_m \cdot (e^2 - e)^n = z_m \cdot (e^2 - e)^m \cdot (e^2 - e)^{n-m} = z_0 \cdot (e^2 - e)^{n-m} = 0$ , because  $e^2 - e$  is in  $D$ .

Now let  $x$  be any element such that  $xD=0$ ;  $x = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$ . Since  $(e^2 - e)^k = e^k \cdot (e - 1)^k$  has the sum of its coefficients  $= 1 - {}_k C_1 + {}_k C_2 \dots \pm 1 = (1 - 1)^k = 0$ ,  $(e^2 - e)^k$  is in  $D$  for every  $k$ . Take  $k = m + 1$  and consider  $x \cdot (e^2 - e)^{m+1} = 0$ . Since  $\sum_0^m \beta_j z_j \cdot (e^2 - e)^{m+1} = 0$  by (2), we have  $\sum_1^n \alpha_i e^i \cdot (e^2 - e)^{m+1} = 0$ . This is possible only if all the  $\alpha_i$  are zero, and therefore  $x = \sum_0^m \beta_j z_j$ . If there is a  $\beta_m \neq 0$ , with  $m > 0$ , consider  $0 = x \cdot (e^2 - e)^m = \sum_0^m \beta_j z_j \cdot (e^2 - e)^m = \beta_m z_0$ , by (2) and (1). Since this is impossible, all the  $\beta_m$ 's,  $m > 0$ , must be zero, and  $x = \beta_0 z_0$ ,  $x$  is in  $Q$ .

$$(2) \{x: xQ=0\} = D.$$

PROOF. As in the proof of (1) it is clear that  $D \cdot Q = 0$ . To show that  $\{x: xQ=0\} \subseteq D$ , let  $x$  be any element such that  $xQ=0$ ,  $x = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$ . Since  $xQ=0$ ,  $xz_0=0$  and therefore  $\sum_1^n \alpha_i e^i \cdot z_0 = z_0 \cdot \sum_1^n \alpha_i = 0$ . Thus  $\sum_1^n \alpha_i = 0$ , and  $x$  is in  $D$ .

$$(3) A - D \text{ is a field.}$$

PROOF. Let  $x$  be any element in  $A$ ,  $x = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$ . Then

$$\begin{aligned} x &= \sum_1^n \alpha_i \cdot e + \left( - \sum_2^n \alpha_i \cdot e + \sum_2^n \alpha_i \cdot e^i \right) + \sum_0^m \beta_j z_j \\ &= \alpha \cdot e + g(e) + \sum_0^m \beta_j z_j, \end{aligned}$$

where  $\alpha = \sum_1^n \alpha_i$  and where  $g(e) + \sum_0^m \beta_j z_j$  is in  $D$ . Thus  $x = \alpha e + d$ , with  $d$  in  $D$ . Therefore  $A - D$  is simply the set of all  $\alpha e + D$ ,  $\alpha$  in  $F$ .

Since  $e - e^2$  is in  $D$ ,  $\alpha e \cdot e = \alpha \cdot e^2 \equiv \alpha \cdot e$  (modulo  $D$ ). Therefore  $e + D$  is the unity element of  $A - D$ . Also  $(\alpha e + D)(\alpha^{-1} \cdot e + D) = e^2 + D = e + D$  and therefore  $A - D$  is a field.

(4) If  $d$  is in  $D$ , not in  $Q$ , there exists an element  $d'$  in  $D$ , not in  $Q$ , such that  $dd' = z_0$ .

PROOF. Let  $x$  be any element in  $D$ ,  $x$  not in  $Q$ . Then  $x = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$ , with  $\sum_1^n \alpha_i = 0$ . If all the  $\alpha_i$ 's are zero,  $x = \sum_0^m \beta_j z_j$ , with  $\beta_m \neq 0$ ,  $m > 0$ . Let  $x' = \beta_m^{-1} \cdot (e^2 - e)^m$ . Then  $x \cdot x' = \sum_0^m \beta_j z_j \cdot \beta_m^{-1} \cdot (e^2 - e)^m = \beta_m z_m \beta_m^{-1} \cdot (e^2 - e)^m$ , by (2) and this is  $= z_0$ , by (1). Clearly  $x'$  is in  $D$  and not in  $Q$ .

Therefore we assume that not all of the  $\alpha_i$ 's are zero,  $\alpha_n \neq 0$ . Consider the following set of elements of  $F$ :

$$\begin{aligned} \gamma_0 &= \sum_1^n \alpha_i, & \gamma_1 &= \sum_1^n {}_i C_1 \cdot \alpha_i, & \gamma_2 &= \sum_2^n {}_i C_2 \cdot \alpha_i, \dots \\ \gamma_s &= \sum_s^n {}_i C_s \cdot \alpha_i, \dots, & \gamma_n &= \sum_n^n {}_i C_n \cdot \alpha_i = \alpha_n. \end{aligned}$$

Thus not all of the  $\gamma_s$  are zero and there exists an integer  $r$  such that  $\gamma_0 = \gamma_1 = \dots = \gamma_{r-1} = 0$ ,  $\gamma_r \neq 0$ . Let  $x' = \gamma_r^{-1} \cdot z_r$  and compute

$$\begin{aligned} \gamma_r \cdot x \cdot x' &= x \cdot z_r = \sum_1^n \alpha_i e^i \cdot z_r + \sum_0^m \beta_j z_j \cdot z_r = \sum_1^n \alpha_i e^i \cdot z_r \\ &= \sum_{i=1}^n \sum_{j=0}^r \alpha_i \cdot {}_i C_j \cdot z_{r-j} \end{aligned}$$

where  ${}_i C_0$  is taken to be 1 and where  ${}_i C_j$ , with  $i < j$ , is taken to be 0. Then

$$\gamma_r \cdot x \cdot x' = z_r \cdot \gamma_0 + z_{r-1} \cdot \gamma_1 + \dots + z_1 \gamma_{r-1} + z_0 \gamma_r = z_0 \gamma_r.$$

Therefore  $x \cdot x' = z_0$ . This establishes (4).

To see that  $D$  is the set of all divisors of zero it is sufficient to show that every divisor of zero is in  $D$ , since we already know that  $D \cdot z_0 = 0$ . Thus let  $x$  be any divisor of zero. Then for some  $y$ ,  $xy = 0$ . If  $x$  is not in  $D$ , then  $y$  must be in  $D$ , because  $A - D$  is a field. If  $y$  is in  $Q$ ,  $y = \alpha z_0$ , then  $x \cdot z_0 = 0$  and  $xQ = 0$ . By (2),  $x$  is in  $D$ . If  $y$  is not in  $Q$ , then by (4), there exists an element  $y'$  in  $D$  such that  $yy' = z_0$ . Then  $xyy' = xz_0 = 0$ ,  $xQ = 0$ ,  $x$  is in  $D$  by (2).

Therefore by Theorem a,  $A$  is a subdirectly irreducible ring.

We must finally prove that  $A$  does not possess a unity element. If  $f$  is a unity element then it must be congruent to  $e$  modulo  $D$ ,  $f = e + d$ . Since  $e$  is not a unity element ( $e \cdot z_1 = z_1 + z_0 \neq z_1$ ),  $d \neq 0$ . Then there

exists an element  $d'$  such that  $dd' = z_0$  and therefore  $fd' = ed' + z_0 = d'$ . That is,  $ed' - d' = -z_0$ . Now  $d = \sum_1^n \alpha_i e^i + \sum_0^m \beta_j z_j$ , with  $\sum_1^n \alpha_i = 0$ . If all the  $\alpha_i$  are zero, then  $d'$  can be taken  $= -\beta_m^{-1} \cdot (e^2 - e)^m$ , (see proof of (4)). Consequently  $e \cdot \beta_m^{-1} \cdot (e^2 - e)^m - \beta_m^{-1} \cdot (e^2 - e)^m = -z_0$ ,  $(e^2 - e)^m \cdot (e - 1) = -\beta_m z_0$ . Multiplying by  $e \cdot z_{m+1}$  this becomes  $(e^2 - e)^{m+1} \cdot z_{m+1} = z_0 = -\beta_m z_0 z_{m+1} \cdot e = 0$ , which is impossible.

On the other hand if the  $\alpha_i$  are not all zero then  $d'$  can be taken  $= \gamma_r^{-1} \cdot z_r$ , (see proof of (4)), and we have  $e \cdot \gamma_r^{-1} \cdot z_r - \gamma_r^{-1} \cdot z_r = -z_0$ . Therefore  $z_r - ez_r = \gamma_r z_0 = z_r - z_r - z_{r-1} = -z_{r-1}$ . This is false unless  $r = 1$ ,  $d' = z_1$ ,  $f = e + e - e^2 + \sum_0^m \beta_j z_j$ . Then  $f \cdot z_2 = 2e \cdot z_2 - e^2 z_2 = 2z_2 + 2z_1 - z_2 - 2z_1 - z_0 = z_2 - z_0 \neq z_2$ . Therefore  $A$  has no unity element.

We might point out that for subdirectly irreducible rings of type  $(\gamma)$ ,  $D$  is a maximal regular ideal and therefore contains  $J$ , the Jacobson radical. In this particular example,  $J = N$ , since  $A - N$  is the set of all finite sums  $\sum_1^n \alpha_i e^i$  and this set has zero Jacobson radical.

Since rings of type  $(\beta)$  and  $(\gamma)$  seem to have such similar properties, one might expect a more intimate relationship between them. However the  $N$  of a ring of type  $(\gamma)$  is not necessarily subdirectly irreducible, as the main example above proves.

One can however show that every ring of type  $(\beta)$  is contained in a ring of type  $(\gamma)$ . The following theorem was pointed out to the author by Professor McCoy. It can be proved for the noncommutative case, and shows further that a ring of type  $(\gamma)$  without a unity element is contained in a ring of type  $(\gamma)$  with a unity element.

**THEOREM 4.** *Every subdirectly irreducible ring without a unity element, can be embedded in a subdirectly irreducible ring with a unity element.*

As Professor McCoy pointed out, the proof of this theorem follows almost immediately from Lemma 5 in the paper, *On the theory of radicals in a ring*, by M. Nagata, Journal of the Mathematical Society of Japan vol. 3 (1951) pp. 330-344.

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