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A NOTE ON FINITE UNIONS OF IDEALS AND SUBGROUPS

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1. **Introduction.** From the theory of ideals in a *commutative* ring R it follows easily that an ideal I of R is contained in the (set-theoretic) union of a finite number of prime ideals P_i ($i = 1, 2, \cdots, n$) of R if and only if I is contained in some *one* of the ideals P_i. A simple direct proof of this will be found in [2, p. 186]. Recently, Behrens [1, p. 171] has shown that the same result holds for the case in which neither commutativity nor associativity is assumed in R.

It is easy to see that if an ideal I of a ring R is contained in the union $A_1 \cup A_2$ of *any two* ideals A_1 and A_2, it must be contained in one of them. For suppose that $I \subseteq A_1 \cup A_2$ and that $I \nsubseteq A_1$. Then there exists an element a_2 of $I \cap A_2$ such that $a_2 \notin A_1$. If $x \in I \cap A_1$, then $x + a_2 \notin A_1$ and therefore $x + a_2 \in A_2$ and $x \in A_2$. That is, $I \cap A_1 \subseteq A_2$ and we have $I \subseteq A_2$. As a matter of fact, this result remains valid if I, A_1, and A_2 are subgroups of an arbitrary group. These observations were pointed out to me by Bailey Brown who also raised several questions about possible generalizations, some of which are partially answered in this note.

The following simple example, due to R. E. Johnson, shows that the above result about the union of two ideals no longer holds when we pass to the union of three ideals. Let R be the ring whose additive group is the direct sum of two two-element cyclic groups, with every products equal to zero. Thus the elements of R may be written as $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$ with componentwise addition modulo 2. Then $A_1 = \{(0, 0), (0, 1)\}$, $A_2 = \{(0, 0), (1, 0)\}$, and $A_3 = \{(0, 0), (1, 1)\}$ are ideals in R, and R is contained in the union of these three

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ideals but not contained in the union of any two of them. However, since $R^2 = 0$, it is clear that R^2 is contained in all three of these ideals. This illustrates the first theorem to be proved below.

THEOREM 1. *Let I and A_i ($i = 1, 2, \cdots, n$) be ideals in a ring R such that $I \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$. If I is not contained in the union of any $n-1$ of the ideals A_i, there exists a positive integer k, depending on n, such that $I^k \subseteq A_1 \cap A_2 \cap \cdots \cap A_n$.*

The second theorem involves only one operation, and applies to subgroups of an arbitrary group G which we shall assume to be written additively. However, we do not assume that G is necessarily abelian.

If I is a subgroup of G, and m a positive integer, we shall find it convenient to use the notation, $mI = \{ma; a \in I\}$. As an example, if G is the additive group of the ring of four elements introduced above, then $2G = 0$ and therefore $2G$ is contained in $A_1 \cap A_2 \cap A_3$. This illustrates our second theorem as follows.

THEOREM 2. *Let I and A_i ($i = 1, 2, \cdots, n$) be subgroups of a group G such that $I \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$. If I is not contained in the union of any $n-1$ of the subgroups A_i, there exists a positive integer l, depending on n, such that $lI \subseteq A_1 \cap A_2 \cap \cdots \cap A_n$.*

2. **Proof of Theorem 1.** We assume that $n \geq 3$ since the hypotheses of either theorem can not be satisfied otherwise. Since $I \subseteq A_1 \cup A_2 \cup \cdots \cup A_n$, then clearly $I = (I \cap A_1) \cup (I \cap A_2) \cup \cdots \cup (I \cap A_n)$. For the purposes of the proof, we shall henceforth assume, as we clearly may without loss of generality, that $I = A_1 \cup A_2 \cup \cdots \cup A_n$. For convenience of notation, we shall also set $K = A_1 \cap A_2 \cap \cdots \cap A_n$.

As a first step we prove the following lemma.

LEMMA. *The intersection of any $n-1$ of the ideals A_i coincides with K.*

It is only necessary to show that this intersection is contained in K. Since I is not contained in the union of any $n-1$ of the ideals A_i, there exists an element a_n of A_n which is not in $A_1 \cup A_2 \cup \cdots \cup A_{n-1}$. If $x \in A_1 \cap A_2 \cap \cdots \cap A_{n-1}$, the element $x + a_n$ of I can not be in A_i ($i = 1, 2, \cdots, n-1$); hence $x + a_n \in A_n$. It follows that $x \in A_n$ and so $x \in K$, as required.

We next establish the theorem for the case in which $n = 3$. Since $A_1 \cup A_2 \subseteq A_1 + A_2$, the relation $I = A_1 \cup A_2 \cup A_3$ implies at once that $I \subseteq (A_1 + A_2) \cup A_3$. Since now I is contained in the union of *two* ideals, we know that it is contained in one of them. However, $I \nsubseteq A_3$, and

hence we have $I \subseteq A_1 + A_2$. Similar arguments show that $I \subseteq A_1 + A_3$ and that $I \subseteq A_2 + A_3$. It follows that

$$I^3 \subseteq (A_1 + A_2)(A_1 + A_3)(A_2 + A_3).$$

But if the right side of this is multiplied out, each term consists of a product containing at least two different ideals A_i $(i = 1, 2, 3)$. The lemma then shows that $I^3 \subseteq K$, and this completes the proof for the case in which $n = 3$.

The proof is now easily completed by induction on n, and we therefore assume the theorem for the case of fewer than n ideals A_i. It follows as above that

(1) $$I \subseteq (A_1 + A_2) \cup A_3 \cup \cdots \cup A_n.$$

Now $I \subseteq A_1 + A_2$ or, by possibly omitting some of the ideals on the right side of (1), we have I contained in the union of some $m < n$ ideals but not in the union of any $m - 1$ of them. Since $I \not\subseteq A_3 \cup \cdots \cup A_n$, one of the remaining ideals must be $A_1 + A_2$. Hence, by our induction hypothesis, there exists a positive integer k_1 such that $I^{k_1} \subseteq A_1 + A_2$. A similar argument applies for each pair of ideals A_i, A_j, $(i \neq j)$. Hence there exists a positive integer k such that

$$I^k \subseteq \prod_{i < j} (A_i + A_j).$$

Now if the product on the right is expanded, it becomes a sum of ideals each of which is a product containing at least $n - 1$ *different* ideals A_i. The lemma then shows that $I^k \subseteq K$, completing the proof.

In general, we have not determined the smallest possible value for the integer k. For $n = 3$, the example given in the introduction shows that $k \geq 2$, while the proof above suggests the value $k = 3$. By other methods, it is possible to show that for $n = 3$, we can always choose $k = 2$; while for $n = 4$, k need not be greater than 6.

REMARK. If one does not assume associativity of multiplication in R, the conclusion of Theorem 1 needs only to be modified to state that some product of I with itself is contained in K. The proof goes through with only trivial changes.

We now give some corollaries of Theorem 1.

COROLLARY 1. *Let I and B_i $(i = 1, 2, \cdots, n)$ be ideals in R with $I \subseteq B_1 \cup B_2 \cup \cdots \cup B_n$, and I not contained in any single ideal B_i. Then there exists a positive integer k such that I^k is contained in at least three of the ideals B_i $(i = 1, 2, \cdots, n)$.*

This follows by taking a minimal set of ideals B_i whose union con-

tains I, and applying the theorem. Such a minimal set must contain at least three ideals since, otherwise, I would be contained in some *one* ideal.

COROLLARY 2. *If B_i $(i=1, 2, \cdots, n)$ are ideals in R such that $B_1 \cup B_2 \cup \cdots \cup B_n$ is an ideal, there exists a positive integer k such that $(B_1 \cup B_2 \cup \cdots \cup B_n)^k$ is contained in some one of the ideals B_i.*

To see this, let $I = B_1 \cup B_2 \cup \cdots \cup B_n$. If $I \subseteq B_i$ for some i, there is nothing to prove. Otherwise we apply the preceding corollary.

Before stating the next corollaries we recall that if the ideal B is semi-prime, then $I^k \subseteq B$ implies that $I \subseteq B$.

COROLLARY 3. *Let I and B_i $(i=1, 2, \cdots, n)$ be ideals in R such that $I \subseteq B_1 \cup B_2 \cup \cdots \cup B_n$, where at least $n-2$ of the ideals B_i are semi-prime. Then I is contained in some one of the ideals B_i.*

COROLLARY 4. *If at least $n-2$ of the ideals B_i $(i=1, 2, \cdots, n)$ are semi-prime, then $B_1 \cup B_2 \cup \cdots \cup B_n$ is an ideal if and only if $B_1 \cup B_2 \cup \cdots \cup B_n \subseteq B_j$ for some j, $1 \leq j \leq n$.*

3. **Proof of Theorem** 2. Again, to simplify the notation, we assume that $I = A_1 \cup A_2 \cup \cdots \cup A_n$, and let $K = A_1 \cap A_2 \cap \cdots \cap A_n$. Since the proof of the lemma involves only additive properties of R and does not use commutativity of addition, the lemma is still valid in the setting of this theorem.

Let m be a fixed integer, $1 < m \leq n-1$, such that for a suitable choice of the positive integer s,

(2) s (intersection of any m subgroups A_i) $\subseteq K$.

The lemma shows that for $m = n-1$ this is true with $s = 1$. We now prove that we can pass from m to $m-1$. Accordingly, we consider any $m-1$ of the subgroups A_i, and for convenience of notation let us choose A_i $(i=1, 2, \cdots, m-1)$. Let $x \in A_1 \cap A_2 \cap \cdots \cap A_{m-1}$, and a_m an element of A_m which is in no A_i $(i \neq m)$. We consider the following elements of I:

(3) $x + a_m, \quad 2x + a_m, \cdots, (n - m + 1)x + a_m.$

If any one of these elements is in A_m, say $tx + a_m \in A_m$, then $tx \in A_m$ and (2) shows that $stx \in K$. Suppose, now, that no element (3) is in A_m. Clearly, no one of these elements can be in $A_1 \cup A_2 \cup \cdots \cup A_{m-1}$, so each is in some one of the subgroups $A_{m+1}, \cdots, A_n$. Since there are $n - m$ of these subgroups and $n - m + 1$ elements (3), at least two of these elements must be in the same subgroup, say A_{m+1}. Suppose,

for example, that $t_1x + a_m \in A_{m+1}$ and $t_2x + a_m \in A_{m+1}$, with $t_1 < t_2 \leqq n - m + 1$. It follows that

$$(t_2x + a_m) - (t_1x + a_m) = (t_2 - t_1)x \in A_{m+1},$$

and hence that $(t_2 - t_1)x \in A_1 \cap A_2 \cap \cdots \cap A_{m-1} \cap A_{m+1}$. Then (2) shows that $s(t_2 - t_1)x \in K$. Since no one of the integers t, t_1, t_2 exceeds $n - m + 1$, we have in every case $s(n - m + 1)! \, x \in K$. Hence, under the assumption (2), we have shown that

(4) $s(n - m + 1)!$(intersection of any $m - 1$ subgroups A_1) $\subseteq K$.

Since we know that (2) is true for $m = n - 1$, it is clear that the reduction from m to $m - 1$ can be continued to yield the existence of a positive integer l such that $lA_i \subseteq K$ ($i = 1, 2, \cdots, n$), and hence that $lI \subseteq K$. The proof is therefore completed.

Actually, the proof shows that l can be chosen to be $2! \, 3! \cdots (n-1)!$ but, except for the case in which $n = 3$, it is not known whether this is the minimum possible choice for l.

COROLLARY 1. *Let I and B_i ($i = 1, 2, \cdots, n$) be subgroups of a group G such that $I \subseteq B_1 \cup B_2 \cup \cdots \cup B_n$. If I is not contained in any single one of the subgroups B_i, then there exists a positive integer l such that lI is contained in at least three of the subgroups B_i ($i = 1, 2, \cdots, n$).*

COROLLARY 2. *If B_i ($i = 1, 2, \cdots, n$) are subgroups of the group G such that $B_1 \cup B_2 \cup \cdots \cup B_n$ is a subgroup, there exists a positive integer l such that $l(B_1 \cup B_2 \cup \cdots \cup B_n)$ is contained in some one of the subgroups B_i.*

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