NOTES ON LINEARLY COMPACT ALGEBRAS

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Let A be a linearly compact ring with ideal neighborhoods of zero, and let N be its radical. Zelinsky shows that [**3**, Theorem 1] A/N is algebraically and topologically isomorphic to a complete direct sum (i.e., a cartesian product) of discrete simple rings with minimum condition. In case A is commutative, then [**3**, Theorem 2] A is algebraically and topologically isomorphic to a complete direct sum of a radical ring and primary rings with units, all the summands being linearly compact. If A is an algebra and the closure of powers of N has zero intersection, he then shows [**4**, Theorem C, p. 320] that A (having the usual properties) satisfies the Wedderburn principal theorem. The restriction of N is needed at two stages: raising of orthogonal idempotents of A/N to orthogonal idempotents of A, and the inductive process of producing the semi-simple part. We propose to show that, if A is commutative, the Wedderburn principal theorem is valid without restriction on N. The problem of raising orthogonal idempotents no longer exists, for idempotents which are orthogonal modulo N are already orthogonal; indeed to each idempotent in A/N there is only one idempotent representative in A. By [**3**, Theorem 2] we can restrict ourselves to primary algebras. Then A/N is a field and we may avail ourselves of the results of field theory to construct the semi-simple part. Our main tool (Lemma 1) is a result of Jacobson [**2**, Theorem 6]. It also follows easily from this that we can raise a countable number of idempotents with no restriction on the radical. We use the terminology of [**3**].

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1. We need the following result which is a slight modification of [**2**, Theorem 6] to topological rings with ideal neighborhoods of zero. Its proof is effectively the same as that of [**2**, Theorem 6].

LEMMA 1. *Let A be a topological ring with ideal neighborhoods of zero. Let I be a closed subring contained in the radical of A. Then, for each $a \in I$, $a=0$ if the closure of aI is I.*

Let A be a linearly compact commutative primary algebra with

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unit over a field K. Then A/N is a field extension of K. Every element u not in the radical has an inverse, for it has an inverse v modulo N and uv has an inverse by the definition of the radical. We assume that A/N is separable over K, i.e., there is a transcendence base $(\alpha_i)_{i\in I}$ such that A/N is separably algebraic over $K((\alpha_i)_{i\in I})$.

THEOREM 1. *Let A be a linearly compact commutative primary algebra with unit over a field K, and let N be its radical. Suppose A/N is separable over K. Then $A=S\oplus N$, where S is a closed subalgebra isomorphic to A/N.*

PROOF. Let $(\alpha_i)_{i\in I}$ be a transcendence base of A/N over K, and let $(a_i)_{i\in I}$ be a set of representatives in A. Then no polynomials in $(a_i)_{i\in I}$ with coefficients in K will take values in N. Hence, A contains the field F generated by $(a_i)_{i\in I}$ over K, and A may be considered as an algebra over F. Therefore, we may assume that A/N is separably algebraic over K.

First consider the case that A/N is finite over K. Then $A/N=K(\theta)$ for some element $\theta\in A/N$. Let a be a representative of θ. Denote by $K[a]$ the polynomial ring in a. $K[a]$ will be the desired algebra if $f(a)=0$, where f is the minimal polynomial of θ. We find such an a as follows. Consider the collection of all subvarieties $a+I$, where a is a representative of θ and I is a closed ideal containing $f(a)$. Partially order this collection by set inclusion: $a+I>a'+I'$ if and only if $a+I\subset a'+I'$. Because of linear compactness, every linearly ordered subset of this collection has a least upper bound. Hence there is a maximal element, $a+I$ say. We claim $f(a)=0$. Suppose $f(a)=n\neq 0$. Since A/N is separably algebraic over K, $f'(a)\not\equiv 0 \pmod N$, where f' denotes the formal derivative of f. Let b be the inverse of $f'(a)$ and $a'=a-bn$. Then $f(a')=f(a-bn)\equiv f(a)-f'(a)bn\equiv 0 \pmod{nI}$. By Lemma 1, $a'+I'>a+I$, where I' is the closure of nI. This contradicts the maximality of $a+I$.

In general, let S be a maximal subfield contained in A. If $A/N\neq S$ then we may extend S to a field $S(a)$ by above paragraph. Hence $A=S\oplus N$. Since N is the unique maximal ideal of A, S is a closed subalgebra of A.

REMARKS. (1) The subfield S is unique if it is algebraic over K.[1] Suppose $A=S\oplus N=S'\oplus N$. Take an $s\in S$, $s=s'+n$ with $s'\in S'$ and $n\in N$. Let f be the minimal polynomial of $s+N=s'+N$. Then $0=f(s)=f(s'+n)=f(s')+f'(s')n+\cdots$. Since $f(s')=0$ and $f'(s')\neq 0$, $n=0$ by Lemma 1. It is clear that if A/N is not algebraic over K then S is no longer unique.

[1] This was suggested to us by Professor Zelinsky.

(2) If A is a primary ring with unit and of characteristic zero (i.e., $kx=0$ implies $x=0$, where k is any positive integer), then Theorem 1 is still valid. For in this case A can be considered as an algebra over the field of rational numbers.

(3) If A/N is finite over K, the commutativity of A enters only in A/N; when we deal with A we only consider elements of the form $a, f(a), f'(a), \cdots$. If only the commutativity of A/N is assumed, we do not know whether S is still unique.

Theorem 1 together with [3, Theorem 2] yields:

THEOREM 2. *Let A be a linearly compact commutative algebra over a field K, with ideal neighborhoods of zero. Then* [3, Theorem 2] *A is algebraically and topologically a complete direct sum of primary algebras with unit and a radical algebra, each summand being linearly compact. Suppose that the quotient algebra of each primary summand over its radical is separable over K. Then A contains a closed subalgebra S such that $A=S\oplus N$ (vector space direct sum), where N is the radical of A. Moreover, S is unique if each quotient algebra is algebraic over K.*

2. The condition that the intersection of powers of N be zero enters into the task of raising idempotents only when we want to raise any number of them. If we are willing to restrict ourselves to a countable number of idempotents then linear compactness alone will do, as is shown in the following lemma. From this we get an analogue of [1, Theorem 1].

LEMMA 2. *Let A be a linear compact ring with ideal neighborhoods of zero, and N its radical. Then a countable number of orthogonal idempotents can be raised to orthogonal idempotents in A.*

PROOF. It suffices to consider two idempotents $\bar{e}, \bar{f}$ in A/N, $\bar{e}\bar{f}=0$. Let e be an idempotent in A representing $\bar{e}$. If a is a representative of $\bar{f}$, then $b=a-ea-ae+eae$ is also a representative, and $eb=be=0$. Let I be the closed principal right ideal generated by $n=b^2-b$. Since $eb=0$, $eI=0$. Consider the collection of subvarieties $a+I$, where $a+N=\bar{f}$, $ea=ae=0$ and I is the closed principal right ideal generated by a^2-a. Partially order this collection by set inclusion. By linear compactness there is a maximal element, $f+I$ say. Suppose $f^2-f\neq 0$. It follows from $(1-2f)^2=1+4(f^2-f)\equiv 1 \pmod{N}$, that $(1-2f)$ has an inverse.[2] Let $n=(f^2-f)(1-2f)^{-1}$, $n\in I$, and let I' be the closure of nI. Then $f'+I'>f+I$, where $f'=f+n$. This contradicts the maximality of $f+I$.

[2] The formal use of 1 is permissible, since it appears only in products with elements of A.

THEOREM 3. *Let A be a linearly compact algebra over a field K with ideal neighborhoods of zero, and let N be its radical. Then* [3, Theorem 1] *A/N is algebraically and topologically isomorphic to a complete direct sum of discrete simple rings with minimal condition. Suppose that the summands are countable in number and that each summand is a total matrix algebra over K. Then there is a closed subalgebra S such that $A=S\oplus N$.*

PROOF. We need only consider one summand. Then we follow the proof of [4, Theorem C, p. 320] summing up the semi-simple parts to get the subalgebra S. Let, therefore, A/N be a total matrix algebra over K and $\bar{e}_{ij}$ $(i, j=1, 2, \cdots, n)$ be a set of matrix units of A/N. The theorem will be established if we can raise $\bar{e}_{ij}$ to a set of matrix units e_{ij} of A. By Lemma 2 we can find all the diagonal elements e_{ii}. It remains to find e_{ij} $(i\neq j)$. It suffices to find all the e_{i1} matching a given set of representatives e_{1i} $(i=2, \cdots, n)$, where $e_{1i}=e_{11}e_{1i}=e_{1i}e_{ii}$. Then put $e_{ij}=e_{i1}e_{1j}$.

We construct, for instance, e_{21} as follows. If a is any representative of $\bar{e}_{21}$ such that $a=e_{22}a=ae_{11}$, and if $ae_{12}=e_{22}$ then $e_{12}a=e_{11}$, for $e_{11}-e_{12}a$ is an idempotent in N. Consider the collection $a+I$, where a is a representative of $\bar{e}_{21}$ with $a=e_{22}a=ae_{11}$ and I is a closed right ideal containing $e_{22}-ae_{12}$. Partially order this collection by set inclusion. By linear compactness there is a maximal element $e_{21}+I$. We wish to show $e_{21}e_{12}=e_{22}$. Suppose $e_{22}-e_{21}e_{12}=n\neq 0$. Then $e'_{21}+(nI)^{-}>e_{21}+I$, where $e'_{21}=2e_{21}-e_{21}e_{12}e_{21}$, contradicting the maximality of $e_{21}+I$.

Let S be the total matrix algebra generated by e_{ij}. Since N is the unique maximal ideal, S is closed. We have $A=S\oplus N$.

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