# NOTE ON A THEOREM OF SCHREIER[1]

A. KARRASS AND D. SOLITAR

Schreier [2] showed (using graph techniques) that if $F$ is a free group and $H$ is a normal subgroup which is finitely generated, then $H$ must be of finite index in $F$. We give an algebraic proof of the following generalization:

THEOREM 1. *Let $F$ be a free group on the free generators $\{a_\nu\}$ and let $H$ be a finitely generated subgroup containing a normal subgroup of $F$. Then $H$ must be of finite index in $F$.*

PROOF. It is well known (see e.g. [1] or [2]) that if $F$ is a free group and $H$ a subgroup of $F$ then

(1) we can select a system of right coset representatives $\{W^*(a_\nu)\}$ of $F$ mod $H$ such that every initial segment of any representative is also a representative (in particular the empty word 1 represents $H$);

(2) furthermore, the words $W^*a_\nu(Wa_\nu)^{*-1}$ (where $(Wa_\nu)^*$ denotes the representative of $Wa_\nu$) define generators for $H$;

(3) and finally, if we denote $W^*a_\nu(Wa_\nu)^{*-1}$ by $s_{W^*,a_\nu}$, then those $s_{W^*,a_\nu}$ which are different from 1 (when looked upon as elements of $F$) are free generators for $H$.

Suppose now $H$ is of infinite index and contains a normal subgroup $(\neq 1)$ of $F$. The latter condition is obviously equivalent to the existence of a (freely reduced) word $U(\neq 1)$ such that $U$ together with all of its conjugates are in $H$. According to (3) above, $H$ will be infinitely generated if there are infinitely many representatives $W^*$ such that $W^*a_\nu$ (for some generator $a_\nu$ of $F$) is not freely equal to $(Wa_\nu)^*$.

Let $K$ be a representative. We first show that the representative of some initial segment $KV$ of $KU$ is such that for some $a_\nu$, $KVa_\nu$ is not freely equal to $(KVa_\nu)^*$. Now $(KU)^*=(KUK^{-1}\cdot K)^*=K^*=K$ (since $KUK^{-1}$ is in $H$). Hence if $KU$ were freely equal to $(KU)^*$, we would have $U$ freely equal to 1. Thus there must be a smallest initial segment $KWa_\nu^\epsilon$, $\epsilon=\pm 1$, which is not freely equal to its representative. Clearly $KW$ then is freely equal to $(KW)^*$. If now $\epsilon=1$, we have $(KW)^*\cdot a_\nu$ is freely equal to $KWa_\nu$ which is not freely equal to its representative. On the other hand if $\epsilon=-1$, then $(KWa_\nu^{-1})^*\cdot a_\nu$ is not freely equal to its representative. For otherwise, $(KWa_\nu^{-1})^*\cdot a_\nu$ is

Received by the editors November 6, 1956.

[1] Work on this note was supported by a grant from the National Science Foundation.

freely equal to its representative $(KW)^*$. But then $(KWa_\nu^{-1})^*$ is freely equal to $(KW)^* \cdot a_\nu^{-1}$ which is freely equal to $KWa_\nu^{-1}$, contrary to the construction of $KWa_\nu^{-1}$. In either case we have the existence of a representative $(KV)^*$, where $V$ is an initial segment of $U$ such that $s_{(KV)^*,a_\nu}$ is a free generator for $H$ of the type (3) above.

It remains to show that there are infinitely many such $(KV)^*$. Indeed if $K$ is a fixed representative, there are only finitely many representatives $L$ such that $(KV_1)^*=(LV_2)^*$, where $V_1$ and $V_2$ are initial segments of $U$. For, suppose $U=V_2V_2'$. Then $L=L^*=(LU)^*=(LV_2V_2')^*=[(LV_2)^*V_2']^*=[(KV_1)^*\cdot V_2']^*$. Since $V_1$ and $V_2'$ are segments of the "constant" $U$, there are only finitely many $V_1$, $V_2'$ and hence only finitely many such $L$. But since $H$ is of infinite index, we have infinitely many representatives $K$ and therefore there must be infinitely many distinct $(KV)^*$ such that $s_{(KV)^*,a_\nu}$ (for some $a_\nu$) is of type (3) above.

THEOREM 2. *A subgroup $H$ of a finitely generated free group $F$ is of finite index in $F$ if and only if $H$ is finitely generated and contains, for some positive integer $d$, $F(X^d)$ (i.e. the subgroup of $F$ generated by all $d$th powers of elements of $F$).*

PROOF. If $H$ is finitely generated and contains $F(X^d)$, Theorem 1 applies and we have $H$ must be of finite index. On the other hand, if $H$ is of finite index and $F$ is finitely generated, then $H$ is finitely generated (see (2) in the proof of Theorem 1). Furthermore, let $n$ be the index of $H$ in $F$. Then for any word $W$ of $F$,

$$1, W, W^2, \cdots, W^n$$

cannot all determine distinct cosets of $H$. Hence, for each word $W$, $W^m$ is in $H$ for some $m$, $1 \leqq m \leqq n$, and so $W^{n!}$ is in $H$ for each $W$. Take $d=n!$.

## REFERENCES

1. W. Hurewicz, *Zu einer Arbeit von O. Schreier*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg vol. 8 (1931) p. 307.

2. O. Schreier, *Die Untergruppen der freien Gruppen*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg vol. 5 (1928) p. 161.

NEW YORK UNIVERSITY