

NOTE ON A THEOREM OF SCHREIER¹

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Schreier [2] showed (using graph techniques) that if F is a free group and H is a normal subgroup which is finitely generated, then H must be of finite index in F . We give an algebraic proof of the following generalization:

THEOREM 1. *Let F be a free group on the free generators $\{a_\nu\}$ and let H be a finitely generated subgroup containing a normal subgroup of F . Then H must be of finite index in F .*

PROOF. It is well known (see e.g. [1] or [2]) that if F is a free group and H a subgroup of F then

(1) we can select a system of right coset representatives $\{W^*(a_\nu)\}$ of $F \bmod H$ such that every initial segment of any representative is also a representative (in particular the empty word 1 represents H);

(2) furthermore, the words $W^*a_\nu(Wa_\nu)^{* - 1}$ (where $(Wa_\nu)^*$ denotes the representative of Wa_ν) define generators for H ;

(3) and finally, if we denote $W^*a_\nu(Wa_\nu)^{* - 1}$ by $s_{W^*a_\nu}$, then those $s_{W^*a_\nu}$, which are different from 1 (when looked upon as elements of F) are free generators for H .

Suppose now H is of infinite index and contains a normal subgroup ($\neq 1$) of F . The latter condition is obviously equivalent to the existence of a (freely reduced) word $U (\neq 1)$ such that U together with all of its conjugates are in H . According to (3) above, H will be infinitely generated if there are infinitely many representatives W^* such that W^*a_ν (for some generator a_ν of F) is not freely equal to $(Wa_\nu)^*$.

Let K be a representative. We first show that the representative of some initial segment KV of KU is such that for some a_ν , KVa_ν is not freely equal to $(KV a_\nu)^*$. Now $(KU)^* = (KUK^{-1} \cdot K)^* = K^* = K$ (since KUK^{-1} is in H). Hence if KU were freely equal to $(KU)^*$, we would have U freely equal to 1. Thus there must be a smallest initial segment KWa_ν^ϵ , $\epsilon = \pm 1$, which is not freely equal to its representative. Clearly KW then is freely equal to $(KW)^*$. If now $\epsilon = 1$, we have $(KW)^* \cdot a_\nu$ is freely equal to KWa_ν , which is not freely equal to its representative. On the other hand if $\epsilon = -1$, then $(KW a_\nu^{-1})^* \cdot a_\nu$ is not freely equal to its representative. For otherwise, $(KW a_\nu^{-1})^* \cdot a_\nu$ is

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freely equal to its representative $(KW)^*$. But then $(KWa_r^{-1})^*$ is freely equal to $(KW)^* \cdot a_r^{-1}$ which is freely equal to KWa_r^{-1} , contrary to the construction of KWa_r^{-1} . In either case we have the existence of a representative $(KV)^*$, where V is an initial segment of U such that $s_{(KV)^*, a_r}$ is a free generator for H of the type (3) above.

It remains to show that there are infinitely many such $(KV)^*$. Indeed if K is a fixed representative, there are only finitely many representatives L such that $(KV_1)^* = (LV_2)^*$, where V_1 and V_2 are initial segments of U . For, suppose $U = V_2 V_2'$. Then $L = L^* = (LU)^* = (LV_2 V_2')^* = [(LV_2)^* V_2']^* = [(KV_1)^* \cdot V_2']^*$. Since V_1 and V_2' are segments of the "constant" U , there are only finitely many V_1, V_2' and hence only finitely many such L . But since H is of infinite index, we have infinitely many representatives K and therefore there must be infinitely many distinct $(KV)^*$ such that $s_{(KV)^*, a_r}$ (for some a_r) is of type (3) above.

THEOREM 2. *A subgroup H of a finitely generated free group F is of finite index in F if and only if H is finitely generated and contains, for some positive integer d , $F(X^d)$ (i.e. the subgroup of F generated by all d th powers of elements of F).*

PROOF. If H is finitely generated and contains $F(X^d)$, Theorem 1 applies and we have H must be of finite index. On the other hand, if H is of finite index and F is finitely generated, then H is finitely generated (see (2) in the proof of Theorem 1). Furthermore, let n be the index of H in F . Then for any word W of F ,

$$1, W, W^2, \dots, W^n$$

cannot all determine distinct cosets of H . Hence, for each word W , W^m is in H for some m , $1 \leq m \leq n$, and so $W^{n!}$ is in H for each W . Take $d = n!$.

REFERENCES

1. W. Hurewicz, *Zu einer Arbeit von O. Schreier*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg vol. 8 (1931) p. 307.
2. O. Schreier, *Die Untergruppen der freien Gruppen*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg vol. 5 (1928) p. 161.

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