THE NONEXISTENCE OF A CERTAIN TYPE OF SIMPLE GROUPS OF ODD ORDER

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1. We consider a finite group G, satisfying the following condition (W):

(W) *The centralizer of any element* $\neq 1$ *of* G *is abelian.*

L. Weisner [**8**] has studied finite groups with this condition (W), and proved that such groups are either solvable or simple. The problem of determining the possible types of simple groups satisfying the condition (W) has not been attacked until quite recently. A few years ago working independently G. E. Wall and the author proved that a nonabelian simple group of *even* order satisfying the condition (W) is isomorphic with the linear fractional group $LF(2, 2^{\mu})$ over a finite field of characteristic 2. The proof of this theorem will be given elsewhere (see Brauer-Suzuki-Wall [**2**]).

The purpose of this note is to show that the order of a nonabelian simple group satisfying (W) must be even. Hence the linear fractional groups $LF(2, 2^{\mu})$ are the only simple groups which satisfy the condition (W).

As a corollary to this result, we can show the nonexistence of the Redéi group of odd order (cf. Redéi [**6**]). Here by a Redéi group we shall mean a nonabelian simple group G such that every proper subgroup of a maximal subgroup of G is abelian. Together with the result by Redéi, we can now conclude that the only Redéi group is the alternating group on 5 letters. A generalization of Redéi's theorem will be considered in the final section.

2. If A is a maximal abelian subgroup of a group satisfying the condition (W) of the preceding section, then A satisfies the following property:

A is the centralizer of every element $\neq 1$ *of* A.

This section will be devoted to a general study of an abelian subgroup of any finite group satisfying the above property. Brauer and Fowler have considered such an abelian subgroup and obtained most of the results in this section (cf. [**1**]).

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Let N be the normalizer of A and

$$n = [A:e] \quad \text{and} \quad l = [N:A].$$

If a conjugate subgroup $\sigma A\sigma^{-1}$ of A contains an element $\rho\neq 1$ of A, then the centralizer $N(\rho)$ of ρ contains both A and $\sigma A\sigma^{-1}$. From the assumption on A, $N(\rho)=A$. Hence $\sigma A\sigma^{-1}\subseteq A$. In particular we conclude the following proposition. Two elements of A are conjugate to each other in G if and only if they are conjugate in N. Since each element $\neq 1$ of A has exactly l conjugate elements in N, l is a divisor of $n-1$. Thus

$$w = (n-1)/l$$

is an integer and G has exactly w conjugate classes $C_1, C_2, \cdots, C_w$ which contain an element $\neq 1$ of A. It is easy to see that N has exactly w irreducible characters $\theta_1, \cdots, \theta_w$ of degree l. They are the induced characters induced by nonprincipal linear characters of A and hence vanish on elements of N not in A. Since conjugate classes $C_i\cap A$ of N are special (cf. [7, Lemmas 4 and 5]), we may consider the "exceptional" characters $\Theta_1, \cdots, \Theta_w$ of G associated with $\theta_1, \cdots, \theta_w$ in case $w\geqq 2$ [7, Lemma 5]. In fact these exceptional characters are defined by the following property:

If we consider the induced characters θ_i^* of G, the decomposition of θ_i^* into irreducible components takes the form

$$\theta_i^* = \epsilon\Theta_i + \Delta$$

where $\epsilon=\pm 1$, Δ is a (generally reducible) character of G (or $\Delta=0$ identically) and ϵ, Δ are independent of i.

We call these characters $\Theta_1, \cdots, \Theta_w$ the *exceptional* characters associated with A, or simply A-characters. The exceptional characters are defined only when $w\geqq 2$. Hence we assume $w\geqq 2$ in the rest of this section.

Exceptional characters satisfy the following properties. Let D be the set of elements in G not conjugate to any element $\neq 1$ of A.

(I) $\Theta_i(\sigma)=\Theta_j(\sigma)$ on $\sigma\in D$ for any pair (i, j). In particular A-characters have the same degree.

This may be proved by using the fact that $\theta_i^*(\sigma)=\theta_j^*(\sigma)$ on $\sigma\in D$.

(II) The exceptional characters are linearly independent on $C_1, \cdots, C_w$.

If $\sum_{i=1}^{w} a_i\Theta_i(\sigma)=0$ for $\sigma\in C_j$ $(j=1, 2, \cdots, w)$, then using (I) we conclude

$$\left(\sum_{i=1}^{w} a_i\Theta_i\right)(\Theta_k - \Theta_l) = 0$$

on G for any pair k, l $(1 \leqq k,\ l \leqq w)$. The orthogonality relations for group characters yield now $a_k = a_l$. Hence $\sum_{i=1}^{w} \Theta_i(\sigma) = 0$ for $\sigma \in C_j$. On the other hand we have

$$\sum_{i=1}^{w} \Theta_i(\sigma) = \epsilon \sum_{i=1}^{w} \theta_i^*(\sigma) - \epsilon w \Delta(\sigma).$$

Hence $\sum_{i=1}^{w} \Theta_i(\sigma) = 0$ implies $w\Delta(\sigma) + 1 = 0$ which is impossible ($\Delta(\sigma)$ is an algebraic integer and $w \geqq 2$).

(III) If B is another abelian subgroup of G satisfying the same condition as A, namely satisfying the condition that B is the centralizer of any element $\neq 1$ of B, and if B is not conjugate to A in G, then $\Theta_1, \cdots, \Theta_w$ are nonexceptional for B.

This is an easy consequence of (I) and (II). If Θ_i is a B-character, every Θ_k $(k = 1, 2, \cdots, w)$ is also a B-character, since $\Theta_i = \Theta_k$ on B by (I). This gives a contradiction to (II) ((II) applied to B).

Let θ_0 be the character of the regular representation of N/A. Then θ_0 is the induced character of N induced by the principal character of A. If θ_0^* is the induced character of G induced by θ_0, then $\theta_0^*(\sigma) = \theta_0(\sigma)$ for $\sigma \neq 1$ of A. Hence

$$\sum_{\sigma \in G} \theta_0^*(\sigma)(\Theta_i(\sigma^{-1}) - \Theta_j(\sigma^{-1}))$$
$$= \epsilon [G:N] \sum_{\sigma \in A} \theta_0(\sigma)(\theta_i(\sigma^{-1}) - \theta_j(\sigma^{-1})) = 0.$$

This implies the following proposition.

(IV) θ_0^* contains $\Theta_1, \cdots, \Theta_w$ with the same multiplicity.

θ_0^* and θ_i^* take the same value on $\sigma \in D$. Hence using (IV) we see that

$$(*) \qquad \theta_0^* - \theta_i^* = 1 - \epsilon \Theta_i + a \sum_{k=1}^{w} \Theta_k + \sum_{\mu} x_\mu X_\mu$$

is a linear relation of irreducible characters, which vanishes on every $\sigma \in D$. Here 1 is the principal character and the X's are the nonprincipal, nonexceptional characters of G. It follows from the definition of the induced characters θ_0^* and θ_i^* that

$$(1/g) \sum_{\sigma \in G} \left| \theta_0^*(\sigma) - \theta_i^*(\sigma) \right|^2 = 1 + l.$$

Hence using the orthogonality relations we conclude that

$$(**) \qquad l = a^2(w - 1) + (a - \epsilon)^2 + \sum x_\mu^2.$$

This relation is essential in the proof of our main result.

The coefficients a and x_μ in the relation (*) can be obtained in a second manner. Consider a linear character ξ of A. Then the character ξ^* of G induced by ξ coincides with one of θ_i^* $(i=0, 1, \cdots, w)$. Suppose X is a nonexceptional character of G, then X is contained in all θ_i^* $(i>0)$ with the same multiplicity, say α. If the multiplicity of X in θ_0^* is β, $\theta_0^*-\theta_i^*$ contains X with the multiplicity $\beta-\alpha$. Hence $x_\mu=\beta-\alpha$ if $X=X_\mu$. By the reciprocity law of Frobenius (cf. [**3**, §246]) the restriction X/A of X to A satisfies the equation

$$X/A = \beta\xi_0 + \alpha \sum \xi$$

on A, where ξ_0 is the principal character of A and the summation of the second term ranges over all nonprincipal irreducible characters. Since A is abelian, every irreducible character of A is linear. Hence comparing degrees in both sides we get

$$Dg(X) = \beta + \alpha(n-1) \equiv \beta - \alpha \pmod{n}.$$

The orthogonality relation yields now $\xi_0+\sum\xi=0$ for any element $\sigma\neq 1$ of A. Hence

$$X(\sigma) = \beta - \alpha \quad \text{on} \quad \sigma \neq 1 \in A.$$

Thus we get the following proposition.

(V) If X is a nonexceptional character, then X takes a rational integral value on $C_1, \cdots, C_w$. Actually if X is contained in (*) with the multiplicity x, then

$$X(\sigma) = x \quad \text{for} \quad \sigma \in C_i$$

and this multiplicity x is characterized by the relations

$$Dg(X) \equiv x \pmod{n} \quad \text{and} \quad |x| \leqq (n-1)/2.$$

In particular X vanishes on $C_1, \cdots, C_w$ if and only if the degree $D_g(X)$ is divisible by n.

The relation $|x|\leqq(n-1)/2$ is a consequence of the equation (**), since $|x|\leqq x^2\leqq l\leqq(n-1)/2$ (we have assumed $w\geqq 2$).

In particular we have

(VI) If X and Y are nonexceptional characters of the same degree then X and Y are contained in (*) with the same multiplicity.

A similar consideration may be applied to the A-characters. Let $\Theta=\Theta_i$ be an A-character. The character θ_i is a sum of l linear characters of $A:\theta_i=\sum'\xi'$. If a linear character ξ of A is not in θ_i, then the induced character ξ^* of G is one of $\theta_1^*, \cdots$, but not θ_i^*. Let α be the multiplicity of Θ in θ_j^* $(j\neq i)$. Then α is independent of j and the multiplicity of Θ in θ_i^* is $\alpha+\epsilon$. Hence if β is the multiplicity of Θ in θ_0^*, then Θ is contained in $\theta_0^*-\theta_i^*$ with the multiplicity $\beta-\alpha-\epsilon$ and

$$\Theta/A = \beta\xi_0 + \alpha \sum \xi + \epsilon \sum{}' \xi'$$

where the third summation extends over l linear characters ξ' contained in θ_i. Hence if $\sigma\neq 1$ is an element of A,

$$\Theta_i(\sigma) = \beta - \alpha + \epsilon\theta_i(\sigma).$$

We have now the following proposition.

(VII) $$\sum_{\sigma\neq 1,\sigma\in A} |\Theta_i(\sigma)|^2 \geqq l(n-l).$$

From the above consideration we get $\Theta_i(\sigma)=\epsilon\theta_i(\sigma)+a$ for $\sigma\in A$, $\sigma\neq 1$, where a is a rational integer. Hence

$$\begin{aligned}\sum_{\sigma\neq 1,\sigma\in A} |\Theta_i(\sigma)|^2 &= \sum_{\sigma\neq 1,\sigma\in A} (\epsilon\theta_i(\sigma)+a)(\epsilon\theta_i(\sigma^{-1})+a)\\ &= \sum_{\sigma} |\theta_i(\sigma)|^2 + \epsilon a \sum_{\sigma} (\theta_i(\sigma)+\theta_i(\sigma^{-1})) + a^2(n-1)\\ &= nl - l^2 - 2\epsilon al + a^2(n-1) \geqq l(n-l),\end{aligned}$$

since $a^2w-2\epsilon a\geqq 0$ for $w\geqq 2$ and any integer a.

3. Let G be a nonabelian simple group satisfying the condition (W) of the first section. Then each maximal abelian subgroup of G is a subgroup of the type discussed in the §2. If A and B are two maximal abelian subgroups of G, and if $A\cap B\neq e$, then we can take an element $\sigma\neq 1$ of $A\cap B$. Then both A and B are contained in the centralizer of σ which has been assumed to be abelian. Hence by maximality of A and B we conclude $A=B$. Thus two distinct maximal abelian subgroups have only the identity in common: in other words maximal abelian subgroups make up a partition of G.

The totality of maximal abelian subgroups are divided into classes of conjugate subgroups. Select one representative from each conjugate class and let $A_1, A_2, \cdots, A_s$ be the complete system of such representatives. We shall denote by N_i the normalizer of A_i in G and let

$$[A_i:e] = n_i \quad\text{and}\quad [N_i:A_i] = l_i \qquad (i = 1, 2, \cdots, s).$$

Each A_i is the centralizer of any element $\neq 1$ of A_i, and hence

$$w_i = (n_i-1)/l_i$$

is an integer.

Our object is to show that the order of G is even. By way of contradiction we shall assume that the order g of G is odd. The essential consequence from this hypothesis is the following:

For each i, w_i is an even integer; in particular $w_i \geqq 2$.

We may apply the argument and results in the preceding section for each A_i $(i=1, 2, \cdots, s)$. We have therefore exactly w_i exceptional characters associated with A_i for each i. By (III) of the §2, exceptional characters associated with A_i are not exceptional for A_j $(j \neq i)$. Hence we have $\sum_{i=1}^{s} w_i$ characters of G which are exceptional for some of $A_1, \cdots, A_s$. On the other hand, every element is conjugate to some element of $A_1, \cdots, A_s$ and each A_i contributes exactly w_i conjugate classes of G. Hence G has exactly $1+\sum_{i=1}^{s} w_i$ conjugate classes. By a main theorem of group characters, the total number of distinct irreducible characters of a finite group G is equal to the number of conjugate classes in G. Hence we have obtained all the irreducible character of G except the principal one by taking exceptional characters for $A_1, \cdots, A_s$. This means that every non-principal characters of G is exceptional for some A_i.

Assume that the notation is so chosen that n_s is the smallest integer among the n_i's and in order to simplify the formula we write

$$n_s = n, \qquad l_s = l \quad \text{and} \quad w_s = w.$$

Moreover we assume, in suitably chosen notations, that the degree d of the A_s-characters is divisible by n_i with $i \leqq t$ but not divisible by n_j $(s > t)$. The relation (*) of the preceding section is now written as

$$\Gamma = 1 - \epsilon\Theta_i + a \sum_{k=1}^{w} \Theta_k + \sum_{\mu} x_\mu X_\mu,$$

where the Θ_k's are A_s-characters and the X_μ's are not. By (I) of the §2 A_i-characters $(i<s)$ have the same degree and hence by (VI) they are contained in Γ with the same multiplicity. Let $\{\Theta_j^i\}$ be the totality of A_i-characters $(i=1, \cdots, s-1; j=1, \cdots, w_i)$. Then we can write Γ as

$$\Gamma = 1 - \epsilon\Theta_i + a \sum_{k=1}^{w} \Theta_k + \sum_{\mu=1}^{s-1} x_\mu \sum_{j=1}^{w_\mu} \Theta_j^\mu.$$

Hence Γ will vanish on elements of $A_1, \cdots, A_{s-1}$. Suppose $x_i=0$ for some $i \leqq t$. Then Γ does not contain A_i-characters. Since $i \leqq t$, it follows from (V), that $\Theta_k(\sigma)=0$ and each $\Theta_j^\mu(\sigma)$ is an integer y_μ for $\sigma \neq 1$ of A_i. Hence $\Gamma(\sigma)=0$ is now read as

$$0 = 1 + \sum_{\mu \neq i} x_\mu y_\mu w_\mu$$

which is impossible, since x_μ, y_μ, w_μ are integers and all the w_μ's are even. Hence we conclude that $x_i \neq 0$ for $i=1, \cdots, t$. The relation (**) of the preceding section shows

$$l = a^2(w-1) + (a-\epsilon)^2 + \sum_\mu x_\mu^2 w_\mu$$

$$\geqq 1 + \sum_{i=1}^{t} x_i^2 w_i.$$

Since each x_i $(i=1, \cdots, t)$ is a nonvanishing rational integer we conclude

$$l - 1 \geqq \sum_{i=1}^{t} w_i.$$

The orthogonality relation for $\Theta = \Theta_i$ may be written as

$$g = \sum_{\sigma \in G} |\Theta(\sigma)|^2$$

$$= d^2 + \sum_{i=1}^{s} (g/n_i l_i) \sum_{\sigma \neq 1, \sigma \in A_i} |\Theta(\sigma)|^2.$$

If $i \leqq t$, we have by (V) $\Theta(\sigma)=0$ for $\sigma \neq 1$ of A_i. Thus

$$\sum_{\sigma \neq 1, \sigma \in A_i} |\Theta(\sigma)|^2 = 0 \qquad \text{for } i = 1, 2, \cdots, t.$$

If $t < i < s$, then $\Theta(\sigma)$ is a nonvanishing rational integer by (V). Hence

$$\sum_{\sigma \neq 1, \sigma \in A_i} |\Theta(\sigma)|^2 \geqq n_i - 1 \qquad \text{for } i = t+1, \cdots, s-1.$$

For $i=s$, we have by (VII)

$$\sum_{\sigma \neq 1, \sigma \in A_i} |\Theta(\sigma)|^2 \geqq l(n-l).$$

Hence we conclude that

$$g \geqq d^2 + \sum_{i=t+1}^{s-1} (g/n_i l_i)(n_i - 1) + (g/n)(n-l).$$

On the other hand, every element of G is conjugate to some element of $A_1, \cdots, A_s$. Hence

$$g = 1 + \sum_{i=1}^{s} (g/n_i l_i)(n_i - 1).$$

Hence

$$1 + \sum_{i=1}^{t} (g/n_i l_i)(n_i - 1) + (g/nl)(n - 1) \geqq d^2 + (g/n)(n - l).$$

It follows now

$$\sum_{i=1}^{t} (n_i - 1)/n_i l_i + (n - 1 + l^2)/nl \geqq 1 + (d^2 - 1)/g.$$

By definition $(n_i - 1)/l_i = w_i$, $n_i \geqq n$ for $i = 1, 2, \cdots, t$. Hence

$$\left(\sum_{i=1}^{t} w_i\right) \Big/ n \geqq \sum_{i=1}^{t} (n_i - 1)/n_i l_i.$$

We have already obtained the inequality

$$l - 1 \geqq \sum_{i=1}^{t} w_i.$$

Using these estimations we get

$$(l - 1)/n + (n - 1 + l^2)/nl \geqq 1 + (d^2 - 1)/g.$$

Since we have assumed that G is a nonabelian simple group, d is greater than one as being the degree of a nonprincipal character. Hence

$$(l - 1)/n + (n - 1 + l^2)/nl > 1$$

Since $n - 1 = wl$, we may write this strict inequality in the form

$$l - 1 + w + l > wl + 1, \qquad 2(l - 1) + w(1 - l) > 0$$
$$(l - 1)(2 - w) > 0.$$

This is however impossible, because $l \geqq 1$ and $w \geqq 2$.

Thus we have shown the validity of the following theorem.

THEOREM. *If a nonsolvable group G satisfies the condition* (W) *of the first section, then the order of G must be even.*

Together with the result in [2], we may conclude the following:

THEOREM. *The nonsolvable group G satisfying the condition* (W) *is one of the linear groups $LF(2, 2^\mu)$.*

4. In this section we shall consider a nonabelian simple group G such that every maximal subgroup of G contains only *nilpotent* proper subgroups. Our purpose is to show that such a group is isomorphic with the alternating group on 5 letters.

First of all, we shall show that the Sylow subgroups of G are independent, i.e. their intersection is $\{e\}$. By way of contradiction, we shall assume the maximal p-intersection D is not the unit subgroup. The normalizer $N(D)$ is not nilpotent. Hence $N(D)$ is a maximal subgroup, which is not nilpotent. By assumption, every proper subgroup of $N(D)$ is nilpotent. Hence by a theorem of Iwasawa [**5**], a p-Sylow subgroup T_p of $N(D)$ is cyclic. Since $N(D)$ is maximal, the normalizer of T_p is contained in $N(D)$. It follows now that T_p is a p-Sylow subgroup of G, and is contained in the center of its normalizer. By a theorem of Burnside ([**3**, p. 237]) G is not simple against our assumption. Hence the Sylow subgroups of G are independent.

Next we shall show that any maximal subgroup of G is not nilpotent. If a maximal subgroup M is nilpotent, then M is the normalizer of the center of Sylow subgroup of M. We have already shown that Sylow subgroups are independent. Hence for all prime divisors of the order of G, G is p-normal in the sense of Grün (cf. [**4**]). By a theorem of Grün we have an isomorphism between p-commutator factor groups $G/G'(p)$ and $N/N'(p)$, where N is the normalizer of the center of a p-Sylow subgroup. We may therefore take $M=N$. Then $M/M'(p)\neq e$ for some p and this implies that the commutator subgroup of G is a proper subgroup, which is not the case. Hence M is not nilpotent.

Let M be any maximal subgroup of G. Then M is not nilpotent, but all proper subgroups of M are nilpotent. Since Sylow subgroups of G are independent, the order of M has the form $p^\alpha q$ where p and q are distinct prime numbers. This follows from the result of Iwasawa [**5**]. Furthermore the p-Sylow subgroup S_p of M is a normal subgroup of M. If T_p is the commutator subgroup of S_p, then T_p and a q-Sylow subgroup T_q of M generate a nilpotent subgroup U. We want to show $T_p=e$. By way of contradiction, assume $T_p\neq e$. Let S_q be a q-Sylow subgroup of G containing T_q. If $\sigma\in T_p$, $\sigma S_q\sigma^{-1}\supseteq\sigma T_q\sigma^{-1}.=T_q$. Since the Sylow subgroups of G are independent, $\sigma S_q\sigma^{-1}=S_q$. Let N be a maximal subgroup of G containing the normalizer of S_q. Then $N\supseteq S_q\cup T_p$. If $N=M$, then $S_q=T_q$ and S_q is contained in the center of the normalizer of S_q. This is impossible by a theorem of Burnside (loc. cit.). Hence $N\neq M$. N is again not nilpotent but every proper subgroup of N is nilpotent. Hence by Iwasawa's theorem (loc. cit.) $N=S_q\cup T_p$ and $[T_p: e]=p$; in particular, $S_q\neq T_q$. $U=T_p\cup T_q$ is therefore a cyclic group of order pq and $U=M\cap N$. The normalizer of U is contained in the normalizer of T_p which is M and at the same time is in the normalizer of T_q which is N. Hence the normalizer of U coincides with $U=M\cap N$. Denote by p^α, q^β the orders of S_p and S_q respectively. Then G contains $(g/p^\alpha q)(p^\alpha-1)$ elements with orders a power of p, where $g=[G: e]$. Similarly conjugate subgroups of S_q contain

$(g/q^\beta p)(q^\beta-1)$ elements other than the identity. Since conjugate subgroups of U contain $(g/pq)(pq-p-q+1)$ elements of order pq, we have an inequality:

$$1+(g/p^\alpha q)(p^\alpha-1)+(g/q^\beta p)(q^\beta-1)+(g/pq)(pq-p-q+1)\leqq g,$$

or

$$(1/g)+(1/pq)\leqq(1/pq)((1/p^{\alpha-1})+(1/q^{\beta-1})).$$

Since $\alpha\geqq 2$, $\beta\geqq 2$, we get

$$1<(1/p)+(1/q).$$

This inequality is however impossible. Hence T_p must be the unit subgroup.

This shows that every maximal subgroup of G contains only abelian proper subgroups. It is now easy to verify that the condition (W) of the opening section is satisfied. By our main theorem the order of G must be even. Hence by a theorem of Redéi [6], G is isomorphic with the alternating group on five letters.

THEOREM. *Let G be a nonabelian simple group. If every maximal subgroup of G contains only nilpotent proper subgroups, then G is isomorphic with the alternating group on five letters.*

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