ON QUASI-ORTHOGONAL POLYNOMIALS[1]

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In the course of a proof, A. Rosenthal [2] used a class of polynomials which, upon specialization of certain constants, may be defined by

$$R_n^\nu(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} x^\nu e^{-x^2} \qquad (n, \nu = 0, 1, 2, \cdots). \tag{1}$$

These polynomials can be expressed in terms of the Hermite polynomials as follows:

$$R_n^\nu(x) = 2^{-\nu}\nu! \sum_{k=0}^{[\nu/2]} \frac{H_{n+\nu-2k}(x)}{(\nu-2k)!k!}.$$

From this relation and the well-known orthogonality of the Hermite polynomials, it follows that these polynomials satisfy the "quasi-orthogonality" relations

$$\int_{-\infty}^{\infty} e^{-x^2} R_m^\nu(x) R_n^\nu(x)dx = 0 \text{ for } m \neq n \pm 2j \quad (j = 0, 1, \cdots, [\nu/2]).$$

The preceding admits of an obvious generalization. We will consider real polynomials although, with appropriate notational changes, complex variables could be considered.

Let $p_n(x)$ denote the nth orthonormal polynomial associated with a distribution $d\alpha(x)$ on an interval (a, b). Let k and r be fixed integers, $k \geqq 0$, $r \geqq 1$. Define $p_{-m}(x) = 0$ for $m = 1, 2, \cdots, kr$. Then the polynomials

$$q_n(x) = \sum_{j=0}^{k} a_{n,j} p_{n-jr}(x) \qquad (a_{n,j} \text{ constants, } a_{n,0} \neq 0) \tag{2}$$

clearly satisfy the relations

$$(q_m, q_n) \equiv \int_a^b q_m(x) q_n(x) d\alpha(x) = 0 \text{ for } m \neq n \pm jr \qquad (j = 0, 1, \cdots, k). \tag{3}$$

Presented to the Society, August 24, 1956, under the title *On a class of polynomials related to the Hermite polynomials*; received by the editors October 29, 1956.

[1] This paper is an extract from the author's thesis (Purdue, 1955) which was written under the direction of Professor Arthur Rosenthal.

We will call polynomials satisfying (3), *quasi-orthogonal* polynomials of index (k, r).

It is easily seen that the above is an extension of the concept of quasi-orthogonal polynomials (of order $n+1$) which was introduced by M. Riesz [1] and which played a fundamental role in his solution of the Hamburger moment problem. The polynomials considered by Riesz were of the form $Ap_n(x)+Bp_{n+1}(x)$, that is, of index $(1, 1)$ in the present notation.

THEOREM. *Let $\alpha(x)$ be a nondecreasing function with infinitely many points of increase in (a, b) such that the moments $\int_a^b x^n d\alpha$ exist $(n=0, 1, \cdots)$. Then a set of polynomials $\{q_n(x)\}$, $q_n(x)$ of degree n, satisfy* (3) *if and only if they have the structure given by* (2).

PROOF. We have noted that (2) implies (3). For the converse, we take $k>0$ (since for $k=0$, we have orthogonal polynomials). Then since the orthonormal polynomials, $p_n(x)$, associated with $\alpha(x)$ on (a, b) are uniquely determined (up to a factor of ± 1) [3, §2.2], we can write

$$q_n(x) = \sum_{j=0}^{n} b_{n,j}p_j(x) \qquad (b_{n,n} \neq 0,\ n = 0, 1, \cdots).$$

According to (3), for $n>kr$, $q_n(x)$ is orthogonal to every polynomial of degree $m<n-kr$. Hence, in particular, we have $(q_n, p_m)=b_{n,m}=0$ for $m=0, 1, \cdots, n-kr-1$, and we can write

$$q_n(x) = \sum_{j=0}^{kr} b_{n,n-j}p_{n-j}(x) \qquad (n = 0, 1, 2, \cdots)$$

where, for convenience, we define $b_{n,n-j}=0$ if $n-j<0$.

It thus remains to show that, for $r>1$, $b_{n,n-j}=0$ if $j\neq 0, r, \cdots, kr$. To this end, let $n>0$ be fixed and assume this is true for every $m<n$; that is, assume

$$q_m(x) = \sum_{i=0}^{k} b_{m,m-ir}p_{m-ir}(x) \qquad (m = 0, 1, \cdots, n-1). \tag{4}$$

Now (4) certainly holds for $m=0$. Suppose it failed for $m=n$. Then there exist certain $b_{n,n-ir+j}\neq 0$, $0<j<r$, $n-ir+j\geqq 0$. Let $n-sr+t$ be the least integer such that $b_{n,n-sr+t}\neq 0$. Then necessarily $n-sr+t\geqq 0$ and (4) implies, by virtue of (3):

$$(q_n, q_{n-sr+t}) = b_{n,n-sr+t}\cdot b_{n-sr+t,n-sr+t} = 0.$$

Hence we have a contradiction which establishes (4) for $m=n$.

We next note that quasi-orthogonal polynomials satisfy a three term recurrence formula with polynomial coefficients. For in the preceding notation, we have for $n > kr$

$$q_{n+i}(x) = \sum_{j=0}^{k} a_{n+i,j} p_{n-jr+i}(x) \qquad (i = -1, 0, 1).$$

If we also write the classical recurrence formula for orthogonal polynomials

$$0 = p_{n+1-s}(x) + (\alpha_{n-s}x + \beta_{n-s})p_{n-s}(x) + \gamma_{n-s}p_{n-s-1}(x)$$

for $s = 0, 1, \cdots, kr$, we have a system of $kr+4$ equations from which the $kr+3$ polynomials, $p_m(x)$ $(m = n-kr+1, n-kr, \cdots, n+1)$, can be eliminated. The result of this elimination is a relation of the form

$$A_n(x)q_{n+1}(x) + B_n(x)q_n(x) + C_n(x)q_{n-1}(x) = 0, \tag{5}$$

where $A_n(x)$, $B_n(x)$ and $C_n(x)$ are polynomials in x. By explicitly writing the eliminant of the system, it can be shown that their degrees are at most kr, $kr+1$ and kr, respectively. Further, these are the exact degrees if and only if $a_{n,k} \cdot a_{n-1,k} \neq 0$.

A similar relation holds for $n \leqq kr$, the degrees of $A_n(x)$, $B_n(x)$ and $C_n(x)$ being at most n, $n+1$ and n in this case.

For the polynomials given by (1), in the particular cases $q_{n+\nu}(x) = R_n^\nu(x)$ for $\nu = 2$ and 3, (5) takes the forms

$$\begin{aligned}(2x^2 + n^2 - n)R_{n+1}^3(x) &- 2x(2x^2 + n^2 - n - 2)R_n^2(x) \\ &+ 2n(2x^2 + n^2 + n)R_{n-1}^2(x) = 0,\\ (6x^2 + n^2 - n)R_{n+2}^2(x) &- 2x(6x^2 + n^2 - n - 6)R_{n+1}^3(x) \\ &+ 2(n+2)(6x^2 + n^2 + n)R_n^3(x) = 0.\end{aligned}$$

No explicit expression for the coefficient polynomials was found for general ν (other than by determinants) although it can be shown that $A_n(x)$, $B_n(x)$ and $C_n(x)$ are even, odd and even functions of x, respectively.

References

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2. A. Rosenthal, *On functions with infinitely many derivatives*, Proc. Amer. Math. Soc. vol. 4 (1953) pp. 600–602.

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