and hence lim sup $(M(r)/p(r)) \geqq 1$, which completes the proof of the theorem.

References

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AN ELEMENTARY PROOF OF THE CLOSURE IN L OF TRANSLATIONS OF e^{-x^2}, AND THE BOREL TAUBERIAN THEOREM

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It is well known that the Tauberian Theorem[1] for Borel summation can be deduced easily from the closure of translations of e^{-x^2} in $L(-\infty, \infty)$, this last result being a special case of Wiener's General Tauberian Theorem.[2]

A simple proof of the Littlewood Tauberian Theorem for Abel summation has been given by Karamata[3] by a method which depends on the fact that the closure theorem for the Abel kernel is closely related to the Weierstrass theorem on polynomial approximation to arbitrary functions and can be proved by elementary means. This suggests that it might be of interest to find elementary proofs of the closure theorems, and the associated Tauberian theorems, for other kernels by using their specific properties rather than Wiener's general theorem. We show here that this can be done very simply for the Borel kernel e^{-x^2}.

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[1] G. H. Hardy and J. E. Littlewood, *Theorem concerning the summability of series by Borel's exponential method*, Rend. Circ. Mat. Palermo vol. 41 (1916) pp. 36–53.

[2] N. Wiener, *The Fourier integral and certain of its applications*, Cambridge, 1933.

[3] J. Karamata, *Über die Hardy-Littlewoodsche Umkehrung des Abelschen Stetigkeitssatzes*, Math. Zeit. vol. 32 (1930) pp. 319–320.

Since

$$f(x) = \underset{\alpha\to\infty}{\text{l.i.m.}} \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha(x-y)^2} f(y)dy$$

when $f(x)$ is in L, it is plainly enough to show that, for any positive β, $e^{-(1+\beta)x^2}$ can be approximated in L by linear sums of translations of e^{-x^2}, and this follows at once from the following.

LEMMA. *If* $0<\beta<1$, *and* $\epsilon>0$, *we can define*

$$P(x) = \sum_{k=1}^{K} A_k e^{-(x-a_k)^2} \tag{1}$$

with constants A_k, a_k *so that*

$$\int_{-\infty}^{\infty} \left| e^{-(1+\beta)x^2} - P(x) \right| dx < \epsilon. \tag{2}$$

PROOF. It is plain that

$$P_n(x) = e^{-x^2}\left[2 - \cosh x(2\beta/n)^{1/2}\right]^n$$

is a function of the required type (1). But

$$\begin{aligned} P_n(x) &= e^{-(1+\beta)x^2}\left\{e^{\beta x^2/n}\left[2 - \cosh x(2\beta/n)^{1/2}\right]\right\}^n \\ &= e^{-(1+\beta)x^2}\left[1 + O(x^4/n)\right] \qquad \text{for } |x| \leqq n^{1/2}, \end{aligned}$$

and since $|2-\cosh y| \leqq e^{y^2/2}$ for all y, we have also

$$|P_n(x)| \leqq e^{-x^2(1-\beta)}$$

for all x. Hence, if $n^{1/2}>B>0$,

$$\int \left| e^{-(1+\beta)x^2} - P_n(x) \right| dx \leqq 2\int_{B}^{\infty} \left[e^{-(1+\beta)x^2} + e^{-(1-\beta)x^2}\right]dx + O(B^5/n),$$

and (2) follows by choice of B and n.

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