

3. ———, *Perturbation of normal operators*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 103–110.
4. F. Rellich, *Störungstheorie der Spectralzerlegung*, III, Math. Ann. vol. 116 (1939) pp. 555–570; V, Math. Ann. vol. 118 (1942) pp. 462–484.
5. B. v. Sz.-Nagy, *Perturbations des transformations lineaires fermées*, Acta Univ. Szeged. vol. 14 (1951) pp. 125–137.
6. F. Riesz and B. v. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Budapest, 1953.
7. H. Wielandt, *Pairs of normal matrices with property L*, Journal of Research of the National Bureau of Standards vol. 51 (1953) pp. 89–90.
8. F. Wolf, *Analytic perturbation of operators in Banach spaces*, Math. Ann. vol. 124 (1952) pp. 317–333.

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ON A THEOREM OF MAGNUS

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1. In a recent paper [2],² W. Magnus has shown that analogues of the Fourier inversion and Plancherel theorems hold for matrix-valued functions on the real line R . We propose to show that these theorems actually hold for an arbitrary locally compact Abelian group, and that Magnus's inversion integral (l. c. (1.4)) can be simplified. For all group- and integral-theoretic notation, terms, and facts used here without explanation, see [1].

2. Let G be a locally compact Abelian group, written additively, with character group X . Elements of G will be denoted " s ", " t ", and elements of X by " χ ", with or without subscripts. The differential of Haar measure on G [X] will be denoted dt [$d\chi$] and these measures are to be so chosen that equality obtains in the Fourier inversion theorem [1, p. 143] and Plancherel's theorem [1, p. 145].

2.1. Let U be a continuous n -dimensional unitary representation of G , so that: $U(s+t) = U(s)U(t)$ for all $s, t \in G$; $U(0) = I$; and all coefficients u_{jk} of U are continuous functions on G . Then the reduction theorem states that there exist a unitary matrix V and characters $\chi_1, \dots, \chi_n \in X$ such that

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² Numbers in brackets refer to the bibliography.

$$2.1.1 \quad U(t) = V^{-1} \begin{pmatrix} \chi_1(t) & 0 & \cdots & 0 \\ 0 & \chi_2(t) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \chi_n(t) \end{pmatrix} V$$

for all $t \in G$.

The set $\{\chi_1, \dots, \chi_n\}$ of characters is completely determined by U , although obviously their order is not.

2.2. We now select a symmetric compact neighborhood A of the identity in X having (finite) positive measure. Let $E = E(\chi_1, \dots, \chi_n)$ be the function on X^n that is equal to 1 if $\chi_j \chi_k^{-1} \in A$ for all j, k and is equal to 0 otherwise. It is obvious that

$$2.2.1 \quad E(\chi_1, \dots, \chi_n) = E(\chi_1 \chi, \dots, \chi_n \chi)$$

for all $\chi \in X$.

3. Let $f \in L_1(G)$. We say that Fourier inversion holds for f if $\hat{f} \in L_1(X)$ and $\int_X \hat{f}(\chi) \overline{\chi(s)} d\chi = f(s)$ for all $s \in G$. (Recall that $\hat{f}(\chi) = \int_G f(t) \chi(t) dt$.) It is known [1, p. 143] that Fourier inversion holds if f is in $L_1(G)$ and is also a linear combination of continuous positive definite functions. We can now establish our generalization of Magnus's theorem.

3.1. THEOREM. *Let $F = F(t)$ be an $n \times n$ complex matrix function on G such that $F_{jk} \in L_1(G)$ and Fourier inversion holds for F_{jk} , for all coefficients F_{jk} of F . For every representation as in 2.1, let*

$$3.1.1 \quad F^\wedge(U) = \int_G F(t) U(t) dt.$$

Then for all $s \in G$, the equality

$$3.1.2 \quad \kappa \cdot F(s) = \int_{X^n} F^\wedge(U) U(-s) E(\chi_1, \dots, \chi_n) d\chi_1 \cdots d\chi_n$$

holds, where κ depends only on n and A and $0 < \kappa < \infty$. The integral in 3.1.2 is an n -fold group integral over X^n , in which the integrand vanishes unless $\chi_j \chi_k^{-1} \in A$ for all j and k . The matrix $F^\wedge(U) U(-s)$ depends symmetrically on the characters χ_1, \dots, χ_n associated with U by 2.1.1, and hence the integrand in 3.1.2 is well defined.

PROOF. Let j and k be arbitrary integers, $1 \leq j \leq n$ and $1 \leq k \leq n$. Using 3.1.1 and 2.1.1, we can write the (j, k) th entry of the integral in 3.1.2 as

$$3.1.3 \quad \sum_{\alpha, \beta=1}^n \int_{X^n} \int_G F_{j\alpha}(t) \chi_\beta(t-s) \bar{v}_{\beta\alpha} v_{\beta k} E(\chi_1, \dots, \chi_n) dt d\chi_1 \cdots d\chi_n.$$

(Here and also below we make free use of Fubini's theorem.) For each α and β , the integral in 3.1.3 can be written as

$$3.1.4 \quad \bar{v}_{\beta\alpha} v_{\beta k} \int_{X^n} \hat{F}_{j\alpha}(\chi_\beta) \chi_\beta(-s) E(\chi_1, \dots, \chi_n) d\chi_1 \cdots d\chi_n.$$

To evaluate 3.1.4, suppose for simplicity that $\beta=1$. The integral in 3.1.4, taken over χ_2, \dots, χ_n , will be independent of the value of χ_1 . To see this, we substitute 2.2.1 in 3.1.4 and obtain

$$3.1.5 \quad \begin{aligned} & \int_{X^n} E(\chi_1, \chi_2, \dots, \chi_n) d\chi_2 \cdots d\chi_n \\ &= \int_{X^n} E(\chi_1 \chi, \chi_2 \chi, \dots, \chi_n \chi) d\chi_2 \cdots d\chi_n \\ &= \int_{X^n} E(\chi_1 \chi, \chi_2, \dots, \chi_n) d\chi_2 \cdots d\chi_n. \end{aligned}$$

The last step follows because the Haar integral over X is invariant. Since χ is arbitrary in X , our assertion about 3.1.4 follows. By hypothesis, Fourier inversion holds for $F_{j\alpha}$. Therefore, denoting the value of 3.1.5 by κ (it is easy to see that $0 < \kappa < \infty$), we can write 3.1.4 as

$$3.1.6 \quad \bar{v}_{\beta\alpha} v_{\beta k} \kappa F_{j\alpha}(s).$$

Hence 3.1.3 is equal to $\sum_{\alpha, \beta=1}^n \bar{v}_{\beta\alpha} v_{\beta k} \kappa F_{j\alpha}(s) = \sum_{\alpha=1}^n \delta_{\alpha k} \kappa F_{j\alpha}(s) = \kappa F_{jk}(s)$. This proves the theorem.

3.2. REMARK. The foregoing proof shows that the matrices V and V^{-1} in 2.1.1 play a quite inessential rôle for Fourier inversion. They do not affect the value of the integral 3.1.2, although $\widehat{F}(U)U(-s)$ does depend upon the choice of V . This fact makes it unnecessary to integrate over the unitary group in order to invert the mapping $F \rightarrow \widehat{F}$. Note that Magnus's inversion formula involves integrating over the unitary group.

4. The Plancherel theorem is very easy to establish in the present context. For a matrix M , let M^* be the conjugate transpose of M and let $\text{Tr } M$ be the trace of M .

4.1. THEOREM. *Let $F = F(t)$ be an $n \times n$ complex matrix-valued function on G such that $F_{jk} \in L_1(G) \cap L_2(G)$ for all coefficients F_{jk} of F . Let $\widehat{F}(U)$ be defined as in 3.1.1. Then*

$$\begin{aligned}
 4.1.1 \quad & \kappa \operatorname{Tr} \int_G F(t)F^*(t)dt \\
 & = \operatorname{Tr} \int_{X^n} F^\wedge(U)F^{\wedge*}(U)E(\chi_1, \dots, \chi_n)d\chi_1 \cdots d\chi_n.
 \end{aligned}$$

PROOF. Let $D(\chi_1, \dots, \chi_n)$ denote the diagonal matrix with entries χ_1, \dots, χ_n . For a fixed representation U , write $U = V^{-1}D(\chi_1, \dots, \chi_n)V$. Let $P(t) = VF(t)V^{-1}$, and let $Q(\chi_1, \dots, \chi_n)$ be the matrix $\{\hat{P}_{jk}(\chi_k)\}_{j,k=1}^n$. Then we have

$$\begin{aligned}
 4.1.2 \quad & F^\wedge(U) = \int_G F(t)U(t)dt = V^{-1} \cdot \int_G P(t) \cdot D(\chi_1(t), \dots, \chi_n(t))dt \cdot V \\
 & = V^{-1}Q(\chi_1, \dots, \chi_n)V.
 \end{aligned}$$

It is plain from 4.1.2 that

$$\operatorname{Tr} (QQ^*) = \operatorname{Tr} (F^\wedge F^{\wedge*}).$$

Hence we find

$$\begin{aligned}
 4.1.3 \quad & \operatorname{Tr} = \int_{X^n} F^\wedge F^{\wedge*}(U)E(\chi_1, \dots, \chi_n)d\chi_1 \cdots d\chi_n \\
 & = \operatorname{Tr} \int_{X^n} Q(\chi_1, \dots, \chi_n)Q^*(\chi_1, \dots, \chi_n)E(\chi_1, \dots, \chi_n)d\chi_1 \cdots d\chi_n \\
 & = \sum_{j,k=1}^n \int_{X^n} |\hat{P}_{jk}(\chi_k)|^2 E(\chi_1, \dots, \chi_n)d\chi_1 \cdots d\chi_n.
 \end{aligned}$$

By the usual Plancherel theorem and the argument used in 3.1, we can rewrite the last term in 4.1.3 as

$$4.1.4 \quad \kappa \cdot \sum_{j,k=1}^n \int_G |P_{jk}(t)|^2 dt.$$

Now using again elementary facts about the trace, we can rewrite 4.1.4 as

$$\begin{aligned}
 4.1.5 \quad & \kappa \int_G \operatorname{Tr} [P(t)P^*(t)]dt = \kappa \int_G \operatorname{Tr} [VF(t)F^*(t)V^{-1}]dt \\
 & = \kappa \int_G \operatorname{Tr} [F(t)F^*(t)]dt = \kappa \operatorname{Tr} \int_G F(t)F^*(t)dt.
 \end{aligned}$$

Combining 4.1.2–4.1.5, we have the present theorem.

4.2. REMARK. If some coefficient F_{jk} of F is in $L_2(G)$ but not in

$L_1(G)$, then the integral 3.1.1 does not exist as an absolutely convergent integral. Nevertheless, a theorem analogous to 4.1 follows with equal ease.

5. We note finally the obvious fact that an “isomorphic” theory can be constructed using the integral

$$\widehat{F}(U) = \int_G U(t)F(t)dt$$

and the inversion integral

$$\int_{\mathcal{X}^n} U(-s)\widehat{F}(U)E(\chi_1, \dots, \chi_n)d\chi_1 \cdots d\chi_n.$$

BIBLIOGRAPHY

1. Lynn H. Loomis, *An introduction to abstract harmonic analysis*, New York, Van Nostrand, 1953.
2. Wilhelm Magnus, *A Fourier theorem for matrices*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 880–890. (Also appeared as Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Report No. BR-8 (1954)).

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