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ON A THEOREM OF MAGNUS

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1. In a recent paper [2],[2] W. Magnus has shown that analogues of the Fourier inversion and Plancherel theorems hold for matrix-valued functions on the real line R. We propose to show that these theorems actually hold for an arbitrary locally compact Abelian group, and that Magnus's inversion integral (l. c. (1.4)) can be simplified. For all group- and integral-theoretic notation, terms, and facts used here without explanation, see [1].

2. Let G be a locally compact Abelian group, written additively, with character group X. Elements of G will be denoted "s", "t", and elements of X by "χ", with or without subscripts. The differential of Haar measure on G [X] will be denoted $dt[d\chi]$ and these measures are to be so chosen that equality obtains in the Fourier inversion theorem [1, p. 143] and Plancherel's theorem [1, p. 145].

2.1. Let U be a continuous n-dimensional unitary representation of G, so that: $U(s+t)=U(s)U(t)$ for all s, $t\in G$; $U(0)=I$; and all coefficients u_{jk} of U are continuous functions on G. Then the reduction theorem states that there exist a unitary matrix V and characters $\chi_1, \cdots, \chi_n \in X$ such that

Received by the editors February 16, 1956 and, in revised form, October 1, 1956.

[1] The first-named author is a fellow of the John Simon Guggenheim Memorial Foundation.

[2] Numbers in brackets refer to the bibliography.

2.1.1
$$U(t) = V^{-1}\begin{pmatrix} \chi_1(t) & 0 & \cdots & 0 \\ 0 & \chi_2(t) & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \chi_n(t) \end{pmatrix} V$$

for all $t \in G$.

The set $\{\chi_1, \cdots, \chi_n\}$ of characters is completely determined by U, although obviously their order is not.

2.2. We now select a symmetric compact neighborhood A of the identity in X having (finite) positive measure. Let $E = E(\chi_1, \cdots, \chi_n)$ be the function on X^n that is equal to 1 if $\chi_j \chi_k^{-1} \in A$ for all j, k and is equal to 0 otherwise. It is obvious that

2.2.1
$$E(\chi_1, \cdots, \chi_n) = E(\chi_1\chi, \cdots, \chi_n\chi)$$

for all $\chi \in X$.

3. Let $f \in L_1(G)$. We say that Fourier inversion holds for f if $\hat{f} \in L_1(X)$ and $\int_X \hat{f}(\chi)\overline{\chi(s)}d\chi = f(s)$ for all $s \in G$. (Recall that $\hat{f}(\chi) = \int_G f(t)\chi(t)dt$.) It is known [1, p. 143] that Fourier inversion holds if f is in $L_1(G)$ and is also a linear combination of continuous positive definite functions. We can now establish our generalization of Magnus's theorem.

3.1. Theorem. *Let $F = F(t)$ be an $n \times n$ complex matrix function on G such that $F_{jk} \in L_1(G)$ and Fourier inversion holds for F_{jk}, for all coefficients F_{jk} of F. For every representation as in* 2.1, *let*

3.1.1
$$F^\frown(U) = \int_G F(t)U(t)dt.$$

Then for all $s \in G$, the equality

3.1.2
$$\kappa \cdot F(s) = \int_{X^n} F^\frown(U)U(-s)E(\chi_1, \cdots, \chi_n)d\chi_1 \cdots d\chi_n$$

holds, where κ depends only on n and A and $0 < \kappa < \infty$. The integral in 3.1.2 *is an n-fold group integral over X^n, in which the integrand vanishes unless $\chi_j\chi_k^{-1} \in A$ for all j and k. The matrix $F^\frown(U)U(-s)$ depends symmetrically on the characters $\chi_1, \cdots, \chi_n$ associated with U by* 2.1.1, *and hence the integrand in* 3.1.2 *is well defined.*

Proof. Let j and k be arbitrary integers, $1 \leqq j \leqq n$ and $1 \leqq k \leqq n$. Using 3.1.1 and 2.1.1, we can write the (j, k)th entry of the integral in 3.1.2 as

3.1.3 $$\sum_{\alpha,\beta=1}^{n} \int_{X^n} \int_G F_{j\alpha}(t)\chi_\beta(t-s)\bar{v}_{\beta\alpha}v_{\beta k}E(\chi_1, \cdots, \chi_n)dtd\chi_1 \cdots d\chi_n.$$

(Here and also below we make free use of Fubini's theorem.) For each α and β, the integral in 3.1.3 can be written as

3.1.4 $$\bar{v}_{\beta\alpha}v_{\beta k}\int_{X^n} \hat{F}_{j\alpha}(\chi_\beta)\chi_\beta(-s)E(\chi_1, \cdots, \chi_n)d\chi_1 \cdots d\chi_n.$$

To evaluate 3.1.4, suppose for simplicity that $\beta=1$. The integral in 3.1.4, taken over $\chi_2, \cdots, \chi_n$, will be independent of the value of χ_1. To see this, we substitute 2.2.1 in 3.1.4 and obtain

3.1.5 $$\begin{aligned} &\int_{X^n} E(\chi_1, \chi_2, \cdots, \chi_n)d\chi_2 \cdots d\chi_n \\ &= \int_{X^n} E(\chi_1\chi, \chi_2\chi, \cdots, \chi_n\chi)d\chi_2 \cdots d\chi_n \\ &= \int_{X^n} E(\chi_1\chi, \chi_2, \cdots, \chi_n)d\chi_2 \cdots d\chi_n. \end{aligned}$$

The last step follows because the Haar integral over X is invariant. Since χ is arbitrary in X, our assertion about 3.1.4 follows. By hypothesis, Fourier inversion holds for $F_{j\alpha}$. Therefore, denoting the value of 3.1.5 by κ (it is easy to see that $0<\kappa<\infty$), we can write 3.1.4 as

3.1.6 $$\bar{v}_{\beta\alpha}v_{\beta k}\kappa F_{j\alpha}(s).$$

Hence 3.1.3 is equal to $\sum_{\alpha,\beta=1}^{n} \bar{v}_{\beta\alpha}v_{\beta k}\kappa F_{j\alpha}(s) = \sum_{\alpha=1}^{n} \delta_{\alpha k}\kappa F_{j\alpha}(s) = \kappa F_{jk}(s)$. This proves the theorem.

3.2. REMARK. The foregoing proof shows that the matrices V and V^{-1} in 2.1.1 play a quite inessential rôle for Fourier inversion. They do not affect the value of the integral 3.1.2, although $F^\frown(U)U(-s)$ does depend upon the choice of V. This fact makes it unnecessary to integrate over the unitary group in order to invert the mapping $F \to F^\frown$. Note that Magnus's inversion formula involves integrating over the unitary group.

4. The Plancherel theorem is very easy to establish in the present context. For a matrix M, let M^* be the conjugate transpose of M and let Tr M be the trace of M.

4.1. THEOREM. *Let $F=F(t)$ be an $n \times n$ complex matrix-valued function on G such that $F_{jk} \in L_1(G) \cap L_2(G)$ for all coefficients F_{jk} of F. Let $F^\frown$ (U) be defined as in* 3.1.1. *Then*

$$\kappa \operatorname{Tr} \int_G F(t)F^*(t)dt$$

4.1.1

$$= \operatorname{Tr} \int_{X^n} F^\frown(U)F^{\frown *}(U)E(\chi_1, \cdots, \chi_n)d\chi_1 \cdots d\chi_n.$$

PROOF. Let $D(\chi_1, \cdots, \chi_n)$ denote the diagonal matrix with entries $\chi_1, \cdots, \chi_n$. For a fixed representation U, write $U = V^{-1}D(\chi_1, \cdots, \chi_n)V$. Let $P(t) = VF(t)V^{-1}$, and let $Q(\chi_1, \cdots, \chi_n)$ be the matrix $\{\hat{P}_{jk}(\chi_k)\}_{j,k=1}^n$. Then we have

4.1.2

$$F^\frown(U) = \int_G F(t)U(t)dt = V^{-1}\cdot \int_G P(t)\cdot D(\chi_1(t), \cdots, \chi_n(t))dt\cdot V$$
$$= V^{-1}Q(\chi_1, \cdots, \chi_n)V.$$

It is plain from 4.1.2 that

$$\operatorname{Tr}(QQ^*) = \operatorname{Tr}(F^\frown F^{\frown *}).$$

Hence we find

$$\operatorname{Tr} = \int_{X^n} F^\frown F^{\frown *}(U)E(\chi_1, \cdots, \chi_n)d\chi_1 \cdots d\chi_n$$

4.1.3

$$= \operatorname{Tr} \int_{X^n} Q(\chi_1, \cdots \chi_n)Q^*(\chi_1, \cdots \chi_n)E(\chi_1, \cdots, \chi_n)d\chi_1 \cdots d\chi_n$$
$$= \sum_{j,k=1}^n \int_{X^n} |\hat{P}_{jk}(\chi_k)|^2E(\chi_1, \cdots, \chi_n)d\chi_1 \cdots d\chi_n.$$

By the usual Plancherel theorem and the argument used in 3.1, we can rewrite the last term in 4.1.3 as

4.1.4

$$\kappa\cdot\sum_{j,k=1}^n \int_G |P_{jk}(t)|^2dt.$$

Now using again elementary facts about the trace, we can rewrite 4.1.4 as

4.1.5

$$\kappa\int_G \operatorname{Tr}[P(t)P^*(t)]dt = \kappa\int_G \operatorname{Tr}[VF(t)F^*(t)V^{-1}]dt$$
$$= \kappa\int_G \operatorname{Tr}[F(t)F^*(t)]dt = \kappa \operatorname{Tr}\int_G F(t)F^*(t)dt.$$

Combining 4.1.2–4.1.5, we have the present theorem.

4.2. REMARK. If some coefficient F_{jk} of F is in $L_2(G)$ but not in

$L_1(G)$, then the integral 3.1.1 does not exist as an absolutely convergent integral. Nevertheless, a theorem analogous to 4.1 follows with equal ease.

5. We note finally the obvious fact that an "isomorphic" theory can be constructed using the integral

$$^\frown F(U) = \int_G U(t)F(t)dt$$

and the inversion integral

$$\int_{X^n} U(-s)^\frown F(U)E(\chi_1, \cdots, \chi_n)d\chi_1 \cdots d\chi_n.$$

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