A NOTE ON THE STONE-WEIERSTRASS THEOREM FOR QUATERNIONS

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A result of M. H. Stone [1, p. 466], which is nicely presented by N. Dunford [2, p. 23], is as follows: Let $A$ be a closed subalgebra of the $B$-algebra $C(X)$ of all continuous real-valued functions on the compact Hausdorff space $X$. Then $A = C(X)$ if and only if $A$ distinguishes between every pair of distinct points of $X$, i.e., for every pair $x_1 \neq x_2$ of points in $X$, there is an $f$ in $X$ such that $f(x_1) \neq f(x_2)$.

If one substitutes the word complex for the word real in the above statement, it becomes false. A well known counter example is obtained by letting $X$ be the set of complex numbers $z$ such that $|z| \leq 1$ and letting $A$ be the subalgebra of functions which are analytic in the interior of $X$.

The purpose of this note is to show that if the word quaternion is substituted for the word real in the above statement, it remains valid. To be specific, let $A$ be a set of continuous quaternion-valued functions which satisfy the following conditions:

1. $A$ is complete.
2. Given a quaternion $q$, the function $f(x) = q$ is in $A$.
3. If $f$ and $g$ are in $A$, then $fg$ and $f + g$ are in $A$.

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If $A$ contains all continuous quaternion-valued functions, it obviously distinguishes between points. Letting $A$ distinguish between points, consider two arbitrary distinct points $x_1$ and $x_2$. Choose an element of $A$ which takes a different value at $x_2$ than at $x_1$. Multiply this function by an appropriate quaternion to obtain a function $f$ such that real part $[f(x_1)] 
eq$ real part $[f(x_2)]$. But the real part of $f$ is $[f - ifi - jjf - kfk]/4$ which is an element of $A$. Therefore, $A$ contains real valued functions which distinguish between points.

Since $A$ is complete, and is closed under multiplication, addition and subtraction, it follows that the set of all real-valued functions in $A$ is also complete and closed under these arithmetic operations. The Stone-Weierstrass Theorem implies that $A$ contains all continuous real-valued functions on $X$. Therefore, $A$ contains all continuous quaternion-valued functions on $X$.

**Bibliography**


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