THE RING $C(X, R)$ CONSIDERED AS A SUBRING OF THE RING OF ALL REAL-VALUED FUNCTIONS

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Conditions under which the ring $C(X, R)$ of all real-valued continuous functions on a completely regular space X determines X have been investigated in a number of papers [1–5]. Gelfand and Kolmogoroff [2] have shown that if X is a compact completely regular space, then the space of maximal ideals in $C(X, R)$ with the Stone topology is homeomorphic to X and hence $C(X, R)$ determines X. The most general result for noncompact spaces has been obtained by F. W. Anderson [1] who has shown that the lattice of all real-valued continuous functions on a topological space determines this space if it is a completely regular space all of whose points are G-delta sets. Since $f(x) \geqq g(x)$ for all x if and only if $f-g$ has square root in $C(X, R)$, a ring isomorphism of $C(X, R)$ implies a lattice isomorphism and Anderson's result is equivalent to the statement that the ring $C(X, R)$ determines X if X is a completely regular space all of whose points are G-delta sets.

In this paper we show that if X is completely regular, then X is topologically determined by the pair of rings consisting of the ring R^X of all real-valued functions on X and its subring $C(X, R)$.

LEMMA 1. *An ideal I in $C(X, R)$ is fixed if and only if there exists a function g in R^X such that $0 \leqq g(x) \leqq 1$ for all x in X, $g(x_0) = 0$ for some x_0 and if f is in I and $-1 \leqq f(x) \leqq 1$ for all x, then $-g(x) \leqq f(x) \leqq g(x)$ for all x.*

PROOF. If all functions in I vanish at some point x_0, then we observe that if we define $g(x) = 1$ for $x \neq x_0$ and $g(x_0) = 0$, then g is the desired function.

Conversely, assume that for an ideal I such a function g exists. If f is continuous, then the function h defined by $h(x) = f(x)$ if $-1 \leqq f(x) \leqq 1$ and $h(x) = 1/f(x)$ otherwise is continuous. If f is in I, then fh is in I, $-1 \leqq (fh)(x) \leqq 1$ for all x, and $(fh)(x) = 0$ if and only if $f(x) = 0$. Hence $-g(x) \leqq (fh)(x) \leqq g(x)$ for all x, $(fh)(x_0) = 0$, and, accordingly, $f(x_0) = 0$. Therefore I is a fixed ideal.

Since every automorphism of R^X preserves the relation $f(x) \leqq g(x)$

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for all x, then *every automorphism of R^X which maps $C(X, R)$ onto itself will map a fixed ideal of $C(X, R)$ onto a fixed ideal of $C(X, R)$.*

THEOREM 1. *If X is a completely regular topological space, then the space of fixed maximal ideals with the Stone topology (i.e. a collection of fixed maximal ideals is closed if and only if their intersection J is equal to the intersection of all fixed maximal ideals which contain J) is homeomorphic to X.*

The proof is similar to the proof [2] that the set of maximal ideals in $C(X, R)$ with the Stone topology is homeomorphic to X if X is compact. One observes that if $C(X, R)$ is a separating class for X (i.e. if $x_1 \neq x_2$, then there is a function f in $C(X, R)$ such that $f(x_1) \neq f(x_2)$) but X is not completely regular, then there is a one-to-one correspondence between X and the space of fixed maximal ideals but the latter space will have fewer closed sets.

From the above results we obtain:

THEOREM 2. *If X and Y are two completely regular spaces and if R^X and R^Y are isomorphic under an isomorphism which maps $C(X, R)$ onto $C(Y, R)$, then X and Y are homeomorphic.*

Lemma 1 and Theorems 1 and 2 also hold if $C(X, R)$ is replaced by the ring $C^*(X, R)$ of all bounded continuous functions on X.

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