COMPACTNESS OF THE STRUCTURE SPACE OF A RING

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Jacobson [1] has shown that a topology may be defined on the set S_A of primitive ideals of any nonradical ring A. With this topology S_A is called the structure space of A. The topology is given by defining closure: if $T=\{p\}$ is a set of primitive ideals then $\overline{T}$ is the set of primitive ideals which contain $\bigcap\{p \mid p\in T\}$. One of Jacobson's results is that if A has a unit then S_A is compact. By working with open sets rather than closure we obtain a simpler proof of this fact and also a new sufficient condition for compactness: every 2-sided ideal of A is finitely generated.

Let $p, q, \cdots,$ be points of S_A (primitive ideals). For each $x\in A$ write (x) for the principal (2 sided) ideal generated by x, and let $U_x=\{p \mid p\not\supset(x)\}$.

LEMMA 1. *$\{U_x\}_{x\in A}$ is a basis of the topology.*

PROOF. Since U_x is the complement of $\{p \mid p\supset(x)\}$ and the latter set is clearly closed, all the U_x are open. Let U be an open subset of S_A, let $F=cU$, and take $p\in U$. Since $F=\overline{F}=\{q \mid q\supset\bigcap_F\}$ and $p\notin F=\overline{F}$, we have $p, \not\supset\bigcap_F$, where $\bigcap_F=\bigcap\{q \mid q\in F\}$. Hence $\exists a\in A$ such that a belongs to the set-theoretic difference $\bigcap_F-p$, and $p\in U_a$. If $q\not\supset(a)$ then $q\not\supset\bigcap_F$, so that $q\notin F$, or $q\in cF=U$. Hence $p\in U_a\subset U$.

THEOREM 1. *If A has a unit then S_A is compact.*

PROOF. We prove that any basic open cover has a finite subcover. Let $\mathfrak{U}=\{U_\mu=\{p \mid p\not\supset(a_\mu)\}\}$ be a basic open cover. Then $S_A=\bigcup_\mu U_\mu$ $=\{p \mid \exists\nu \text{ such that } p\not\supset(a_\nu)\}=\{p \mid p\not\supset\sum_\mu (a_\mu)\}$. Write $I=\sum_\mu (a_\mu)$. In a ring with unit every 2 sided ideal can be imbedded in a primitive ideal [2]. But $\{p \mid p\not\supset I\}=S_A$ exhausts all primitive ideals. Hence $I=A$, so that the unit 1 is in I. Hence $\exists b_1, \cdots, b_n$ in $(a_{\mu_1})+\cdots+(a_{\mu_n})$ such that $1=b_1+\cdots+b_n$, so that $A=(a_{\mu_1})+\cdots+(a_{\mu_n})$ $=I$. But this means that $S_A=\{p \mid p\not\supset\sum_1^n (a_{\mu_i})\}=\bigcup_{i=1}^n U_{\mu_i}$.

The converse of Theorem 1 is false, as is shown by the ring $2J$ of even integers. It is easy to see that the open sets in the structure space of $2J$ are complements of finite sets (the proof in [1] of this same fact for the ring J of all integers carries over *mutatis mutandis*). Hence if $\{G_\alpha\}$ is an open cover, then G_1, say, misses only finitely many points, so that $\{G_\alpha\}$ possesses a finite subcover.

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THEOREM 2. *If every (2 sided) ideal of A is finitely generated then S_A is compact.*

PROOF. With the notation as in Theorem 1, let $\mathfrak{U}=\{U_\mu\}$ be a basic open cover. Then $S_A=\bigcup_\mu U_\mu=\{p \mid p \not\supset \sum_\mu (a_\mu)\}$. Write $I=\sum_\mu (a_\mu)$, and let $b_1, \cdots, b_n$ be a basis of I, so that $I=\sum_i (b_i)$. Each b_i lies in an ideal generated by finitely many of the a_μ, and since there are finitely many b_i, the set $\{b_1, \cdots, b_n\}$ is generated by finitely many a_μ, say $a_{\mu_1}, \cdots, a_{\mu_r}$. That is, $I=\sum_{j=1}^r (a_{\mu_j})$. But this means that $S_A = \{p \mid p \not\supset \sum_{j=1}^r (a_{\mu_j})\} = \bigcup_{j=1}^r U_{\mu_j}$, so that $\mathfrak{U}$ has a finite subcover.

We conclude with a comment which may be regarded as a partial converse to Theorem 2. If S_A is compact then as far as the structure space goes A may as well have been finitely generated. The precise theorem is

THEOREM 3. *If S_A is compact then there exists a finitely generated (2 sided) ideal I of A such that S_A is homeomorphic to S_I.*

PROOF. Here S_I means the structure space of I as a ring. By a theorem of Kaplansky [3] S_I is homeomorphic to $\{p \mid p \not\supset I\}$, for any ideal I of A. We have $S_A=\bigcup_A U_a=\bigcup_{i=1}^n U_{a_i}$ by compactness. Write $I=(a_i)+\cdots+(a_n)$. Then $S_A=\{p \mid p \not\supset I\}$ and this is homeomorphic to S_I by the quoted theorem.

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