A GENERALIZATION OF HILBERT'S NULLSTELLENSATZ

S. A. AMITSUR

1. **Introduction.** The relation between Hilbert's Nullstellensatz and the theory of Jacobson's radical was first pointed out by Krull in [**4**]. This new approach can be utilised to obtain a new proof and an extension of this theorem to the case of polynomials in noncommutative indeterminates.

Let F be a field and let $\overline{F}$ be its algebraic closure. Denote by $F[\xi] = F[\xi_1, \cdots, \xi_n]$ the ring of all polynomials in a set of commutative variables $\xi_1, \cdots, \xi_n$. Let $G = \{g(\xi_1, \cdots, \xi_n)\}$ be a set of polynomials in $F[\xi]$ and let $f(\xi) \in F[\xi]$. The classical form of Hilbert's Nullstellensatz states that: "If $f(\xi)$ vanishes for all the zeros of the polynomials of the set G then some power $f^m(\xi)$ of $f(\xi)$ belongs to the ideal generated by the set G."

By a slight change in the statement of this theorem we can observe possible extensions of this result. To this end we consider the free algebra (with a unit) $F[x_1, \cdots, x_n] = F[x]$ generated by a finite set of *noncommutative* indeterminates $x_1, \cdots, x_n$ over F. Let $G = \{g(x_1, \cdots, x_n)\}$ be a set of polynomials of $F[x]$ and let $I(G)$ be the ideal of $F[x]$ generated by the polynomials of the set G. Denote by $\mathfrak{M}_1$ the ideal generated by the set of commutators $x_ix_j - x_jx_i$, $i, j = 1, \cdots, n$ and consider the following property of a polynomial $f(x) \in F[x]$:

(Z_1) $f(\alpha_1, \cdots, \alpha_n) = 0$ for every set of elements $\alpha_i \in \overline{F}$ for which $g(\alpha_1, \cdots, \alpha_n) = 0$ for all $g(x) \in G$

and the new form of the Nullstellensatz will be:

"If $f(x)$ satisfies (Z_1) then some power of $f(x)$ belongs to the union of the ideals $I(G)$ and $\mathfrak{M}_1$."

The equivalence of the two forms of this theorem follows from the fact that $F[x]/\mathfrak{M}_1 \cong F[\xi_1, \cdots, \xi_n]$, where the ξ_i are commutative algebraically independent elements over F.

In this last form we extend the Nullstellensatz by generalising the property (Z_1). Let $\overline{F}_k$ be the algebra of all square matrices of order n over $\overline{F}$, and let $\mathfrak{M}_n$ be the set of all polynomials $f(x_1, \cdots, x_n) \in F[x]$ for which the relation $f(x_1, \cdots, x_n) = 0$ holds identically in $\overline{F}_n$. It is readily verified that $\mathfrak{M}_n$ is an ideal in[1] $F[x]$ and that $\mathfrak{M}_1$ is the ideal

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[1] Compare with the results of [1] where an infinite number of indeterminates were considered.

generated by the commutators $x_ix_j - x_jx_i$, $i, j = 1, \cdots, n$.

Let R be an algebra over F. A set of elements $(r_1, \cdots, r_n)$ of R is said to be a *zero in* R of the set of polynomials $G = \{g(x_1, \cdots, x_n)\}$ if $g(r_1, \cdots, r_n) = 0$ for all $g(x) \in G$. Consider the following properties of polynomials in $F[x]$:

(Z_k) $f(x_1, \cdots, x_n)$ vanishes for all zeros of G which lie in $\bar{F}_k$.

(Z_∞) $f(x_1, \cdots, x_n)$ vanishes for all zeros of G which lie in primitive rings.

The following extensions of the Nullstellensatz are shown:

THEOREM 1. *If $f(x)$ satisfies (Z_k) then some power $f^m(x)$ belongs to the union of the ideals $I(G)$ and $\mathfrak{M}_k$.*

THEOREM 2. *If $f(x)$ satisfies (Z_∞) then $f^m(x) \in I(G)$ for some integer m.*

Thus, the case $k=1$ is Hilbert's Nullstellensatz.

The proofs are obtained by studying the relations between the Jacobson radical of the quotients $F[x]/(I(G), \mathfrak{M}_k)$, $F[x]/I(G)$ and the properties (Z_k) and (Z_∞) and applying a recent result of the author (Corollary 4 of [2]) which we state for further references as:

LEMMA 1. *The Jacobson radical of a finitely generated algebra over a nondenumerable field is a nil ideal.*

2. **Proofs.** Before proceeding with the proofs of Theorems 1 and 2 we need some lemmas.

Note that the property (Z_k) seems to depend on the underlined field $\bar{F}$ and our first object is to prove its independence. Namely:

LEMMA 2. *If $f(x)$ vanishes for all zeros of G which lie in $\bar{F}_k$ then $f(x)$ vanishes also for all zeros of G which lie in a matrix ring H_k, where H is an arbitrary field-extension of $\bar{F}$.*

Suppose the lemma is not true, then for some extension H of $\bar{F}$ there exists a set of matrices $a_1, \cdots, a_n \in H_k$ such that $g(a_1, \cdots, a_n) = 0$ for all $g(x) \in G$ but $f(a_1, \cdots, a_n) \neq 0$. We shall show that there exists a specialization: $\alpha \to \bar{\alpha}$ of H onto $\bar{F}$ over $\bar{F}$ such that the extended specialization: $a \to \bar{a}$ of H_k onto $\bar{F}_k$ maps the elements a_i on finite matrices $\bar{a}_i \in \bar{F}_k$ and maps $f(a_1, \cdots, a_n)$ onto a nonzero matrix $f(\bar{a}_1, \cdots, \bar{a}_n)$ of $\bar{F}_k$. Clearly, $(\bar{a}_1, \cdots, \bar{a}_n)$ is a zero of G and since $f(x)$ is supposed to satisfy (Z_k) we are led to a contradiction.

To obtain the required specialization we first note that H may be assumed to be a finitely generated extension of $\bar{F}$, since we have to consider only the elements of H which belong to the finite set of matrices $a_1, \cdots, a_n$. Let $\eta_1, \cdots, \eta_r$ be a maximal set of algebraically independent elements of H and let $\zeta_1, \cdots, \zeta_s$ the other generators of H which are algebraic over $\bar{F}(\eta_1, \cdots, \eta_r)$. Before proceeding with the proof we need the following:

LEMMA 3. *Let* $\{\lambda_\nu\}$ *be a finite set of elements in H which are algebraic over* $F(\eta_1, \cdots, \eta_r)$, *then there exists a specialization*: $t \rightarrow \bar{t}$ *of H into* $\bar{F}$ *such that* $\bar{\lambda}_\nu \neq \infty$ *for all* ν, *and if* $\lambda_\nu \neq 0$ *then* $\bar{\lambda}_\nu \neq 0$.

PROOF. The element λ_ν is algebraic over $\bar{F}(\eta)$. Let $\phi_{0\nu}(\eta)\lambda_\nu^r + \cdots + \phi_{r\nu}(\eta) = 0$, where $\phi_{i\nu}(\eta)$ are polynomials in η_j, be a minimum equation of λ_ν over $\bar{F}(\eta)$. If $\lambda_\nu \neq 0$ then $\phi_{0\nu}(\eta)\phi_{r\nu}(\eta) \neq 0$. The required specialization will be any extension of the specialization of $\bar{F}(\eta)$ defined by the mappings: $\eta_i \rightarrow \alpha_i$ where $\alpha_i \in \bar{F}$ are so chosen that $\phi_{0\nu}(\alpha)\phi_{r\nu}(\alpha) \neq 0$, for all ν. One can find such α_i since $\bar{F}$ is algebraically closed and, therefore, infinite. Thus, if $t \rightarrow \bar{t}$ denotes the specialization then $\bar{\lambda}_\nu \neq \infty$ and $\bar{\lambda}_\nu \neq 0$ if $\lambda_\nu \neq 0$. If $\lambda_\nu = 0$ then, obviously, $\bar{\lambda}_\nu = 0$.

We continue now with the proof of Lemma 2. Let $a_i = (\lambda_{jk}^i)$ and $f(a) = \sum f_\nu a_{\nu_1} a_{\nu_2} \cdots a_{\nu_s} = (f_{jk})$, $f_\nu \in F$ and $\lambda_{jk}^i, f_{jk} \in H$. Consider now a specialization of H into $\bar{F}$ which maps nonzero elements of the set (λ_{jk}^i, f_{jk}) onto nonzero elements. Clearly, for this specialization we have, $a \rightarrow \bar{a} \neq \infty$ and $\bar{f}(\bar{a}) = f(\bar{a}) \neq 0, \infty$ since $f(a) \neq 0$. This proves the existence of the required specialization.

An immediate consequence of the preceding lemma is the following:

COROLLARY 1. *If* $f(x)$ *satisfies* (Z_k) *then* $f(x)$ *vanishes also for all zeros of G which lie in simple algebras over F which are of order* $\leqq k^2$ *over their centres.*

Indeed, let $\mathfrak{D}$ be a central simple algebra of order r^2 over its centre Z, where $Z \supset F$ and $r \leqq k$. Let H be a splitting field of $\mathfrak{D}$ containing $\bar{F}$. Since, $\mathfrak{D} \otimes H \cong H_r \subset H_k$, we may consider $\mathfrak{D}$ as a subring of H_k. This means that the zeros of G in $\mathfrak{D}$ are actually in H_k and, thus, the corollary follows by Lemma 2.

For further applications we introduce the following notations: Let $Q_k = (I(G), \mathfrak{M}_k)$ and let J_k/Q_k be the Jacobson radical of the quotient ring $F[x]/Q_k$. Note also that if P/Q_k is a primitive ideal in $F[x]/Q_k$ then P is also primitive in $F[x]$; furthermore, if P is assumed to contain Q_k then the converse is also true. This follows from the isomorphism: $F[x]/P \cong (F[x]/Q_k)/(P/Q_k)$.

LEMMA 4. *If* $f(x)$ *satisfies* (Z_k) *then* $f(x) \in J_k$.

Let $P \supset Q_k$ be a primitive ideal in $F[x]$ and let $\bar{x}_i$ denote the coset of $F[x]/P$ represented by the element x_i. Since every $g(x) \in G$ belongs to Q_k and $Q_k \subset P$, it follows that $g(\bar{x}_1, \cdots, \bar{x}_n) = 0$. This means that $(\bar{x}_1, \cdots, \bar{x}_n)$ is a zero of G in $F[x]/P$. Since $P \supset \mathfrak{M}_k$, one readily verifies as in the proof of Lemma 1 of [1] that $F[x]/P$ satisfies the identities of $\bar{F}_k$. It is known that primitive algebras which satisfy a polynomial identity are central simple algebra of finite order (Kaplansky [3]). It follows readily that a central simple algebra of finite order which satisfies the identities of $\bar{F}_k$ is of order $\leqq k^2$ over its centre. Indeed if $\mathfrak{D}$ is such an algebra and its order over its centre is r^2 and H is a splitting field of $\mathfrak{D}$, then the identities of $\mathfrak{D}$ which are satisfied by $\bar{F}_k$ are satisfied also by $\mathfrak{D} \otimes H \cong H_r$ and H_r satisfies all identities of $\bar{F}_k$ if and only if $r \leqq k$ [1]. Consequently, we obtain that $F[x]/P$ is a central simple algebra of order $r^2 \leqq k^2$ over its centre. Since $(\bar{x}_i)$ is a zero of G in $F[x]/P$, it follows by Corollary 1 that $f(\bar{x}_1, \cdots, \bar{x}_n) = 0$. This is equivalent to the fact that $f(x) \in P$.

Now J_k is the intersection of all primitive ideals P containing Q_k, hence, $f(x) \in J_k$. q.e.d.

Proof of Theorem 1. Assume first that F is nondenumerable, then since $F[x]/Q_k$ is finitely generated, it follows by Lemma 1 and Lemma 3 that if $f(x)$ satisfies (Z_k) then $f(x)$ is nilpotent modulo Q_k.

In the general case let H be a nondenumerable extension of $\bar{F}$., and consider the free algebra $H[x] \supset F[x]$.[2] Let $Q_H(G)$ be the ideal of $H[x]$ generated by the set G and let $\mathfrak{M}_k(H)$ be the set of identities satisfied by the matrix ring H_k. Clearly $Q_H(G) = Q(G) \otimes H$ and we shall show that $\mathfrak{M}_k(H) = \mathfrak{M}_k \otimes H$. It follows now by Lemma 2 and by proof of Theorem 1 for nondenumerable fields that $f(x)$ is nilpotent modulo the ideal $(Q_H(G), \mathfrak{M}_k(H))$. The latter is readily seen to be equal to $(Q(G), \mathfrak{M}_k) \otimes H = Q_k \otimes H$; hence since every power $f^m(x) \in F[x]$, it follows immediately that $f^m(x) \in Q_k \otimes H$ implies that $f^m(x) \in Q_k$.

To complete the proof of Theorem 1 we still have to show that $\mathfrak{M}_k(H) = \mathfrak{M}_k \otimes H$. Let (h_i) be a base of H over $\bar{F}$ and let $\phi(x) \in \mathfrak{M}_k(H)$. Then $\phi(x) = \sum \phi_i(x) h_i$. Note that (h_i) is also a base of H_k over $\bar{F}_k$, hence setting $x_i = a_i \in \bar{F}_k$ we obtain that $0 = \phi(a) = \sum \phi_i(a) h_i$ which implies that $\phi_i(a) = 0$. This proves that $\mathfrak{M}_k(H) \subset \mathfrak{M}_k \otimes H$. On the other hand since $\bar{F}$ is infinite (being algebraically closed), it follows by [1, Lemma 6] that the identities of $\bar{F}_k$ hold also in $H_k (= \bar{F}_k \otimes H)$. That is, $\mathfrak{M}_k \subset \mathfrak{M}_k(H)$. Hence $\mathfrak{M}_k \otimes H \subset \mathfrak{M}_k(H)$, which proves that $\mathfrak{M}_k(H) = \mathfrak{M}_k \otimes H$.

Proof of Theorem 2. Let H be a nondenumerable extension of $\bar{F}$. If $\bar{F}$ is nondenumerable, set $H = \bar{F}$. With the preceding notations, consider the quotient ring $H[x]/Q_H(G)$ and any primitive ideal

[2] We shall identify $H[x]$ with $F[x] \otimes H$.

$P/Q_H(G)$ in it. Denoting by $\bar{x}_i$ the coset of $H[x]/P$ defined by x_i, we obtained similarly to the preceding proof that $(\bar{x}_1, \cdots, \bar{x}_n)$ is a zero of G in $H[x]/P$. The latter is a primitive ring, hence it follows by (Z_∞) that $f(\bar{x}_1, \cdots, \bar{x}_n)=0$ and consequently, $f(x) \in P$. Let $J/Q_H(G)$ be the Jacobson radical of the ring $H[x]/Q_H(G)$, then J is the intersection of all the primitive ideals P of $H[x]$ which contain $Q_H(G)$. From this we conclude that $f(x) \in J$. It follows now, by Lemma 1, that J is a nil ideal; hence $f^m(x) \in Q_H(G)$ for some integer m. Clearly, $Q_H(G) = Q(G) \otimes H$. This together with the fact that $f^m(x) \in F[x]$ yield that $f^m(x) \in Q(G)$. q.e.d.

3. **Modified conditions.** If $Z_k(G)$ denotes the set of all polynomials $f(x)$ satisfying (Z_k) then Lemma 4 states that $Z_k(G) \subseteq J_k$. Generally, $Z_k(G)$ is not identical with J_k but by modifying the condition (Z_k) we can obtain a complete characterization of the polynomials of J_k.

First we introduce the notion of a regular zero. Let $\mathfrak{D}$ be a simple algebra over F and let Z be its centre. A set $(d_1, \cdots, d_n)$ of elements of $\mathfrak{D}$ will be said to be a *regular zero* of G in $\mathfrak{D}$ if the set (d_i) is a zero of G in $\mathfrak{D}$ and the algebra $Z(d_1, \cdots, d_n)$ generated by (d_i) and the centre Z is $\mathfrak{D}$ itself. We replace condition (Z_k) by the following:

(Z_k^R) $f(x_1, \cdots, x_n)$ vanishes for all regular zeros of G in $\bar{F}_r$, $r \leqq k$.

The results of Lemma 2 and Corollary 1 of the preceding section will remain true if one replaces the ordinary zeros by regular zeros. That is:

Lemma 5. *If $f(x)$ satisfies (Z_k^R) then $f(x)$ vanishes also for all regular zeros of G in $\mathfrak{D}$, where $\mathfrak{D}$ is any simple algebra over F such that the order of $\mathfrak{D}$ over its centre is $\leqq k^2$.*

The proof of this lemma is obtained by noting that the proof of Lemma 2 will hold also for regular zeros in H_r, $r \leqq k$. Indeed, if we follow the proof of Lemma 2 we observe that in order to prove that lemma for regular zeros it suffices to show that if (a_i) is a regular zero of G in H_r and $f(a) \neq 0$ then one can choose the specialization, given there, so that $(\bar{a}_i)$ is a regular zero of G in $\bar{F}_r$ and $f(\bar{a}) \neq 0$. To this end we proceed as follows: since (a_i) is a regular zero in H_r, $H(a_1, \cdots, a_n) = H$. Let c_{ik} be the matrices of H_k having the unit in the ith row and kth column, then $c_{ik} = \sum \psi_\nu^{ik} a_{\nu_1} \cdots a_{\nu_s}$. In addition to the requirements $\bar{\lambda}_{jk}^i \neq \infty$ and $\bar{f}_{jk} \neq \infty$ of Lemma 2 we impose also the conditions $\bar{\psi}_\nu^{ik} \neq \infty$ for all i, k. With these conditions satisfied, clearly $(\bar{a})$ is a regular zero in $\bar{F}_k$ and $f(\bar{a}) \neq 0$.

Now let (d) be a regular zero of G in $\mathfrak{D}$ and let $Z \supset F$ be the centre of $\mathfrak{D}$. Let H be a splitting field of $\mathfrak{D}$ containing $\bar{F}$. Since $Z(d) = \mathfrak{D}$ and

$\mathfrak{D}\otimes H\cong H_r$, it follows that $H(d)\cong H_r$ which means that (d) is also a regular zero of G in H_r. The rest of the proof follows as in the proof of Corollary 1.

The following gives the characterization of J_k:

THEOREM 3. *The polynomial* $f(x)\in J_k$ *if and only if* $f(x)$ *satisfies* (Z_k^R).

If $f(x)$ satisfies (Z_k^R) then the fact that $f(x)\in J_k$ follows in the same way as the proof of Lemma 4 with the additional remark that: if P is a primitive ideal in $F[x]$ and $(\bar{x}_i)$ is the zero defined in the proof of Lemma 4, then $(\bar{x}_i)$ is actually a regular zero of G in the ring $F[x]/P$ since $F[\bar{x}_i]=F[x]/P$.

To prove the converse, we consider a regular zero of (a_i) of G in $\overline{F}_r$, $r\leqq k$. Let P be the kernel of the homomorphic mapping of $F[x]$ into $\overline{F}_r$ determined by the correspondence: $x_i\rightarrow a_i$. Since $r\leqq k$, it follows readily that $P\supset\mathfrak{M}_k$ and the fact that (a) is a zero of G implies that $P\supset Q(G)$. Consequently, $P\supset Q_k$. The ideal P is a primitive ideal in $F[x]$: for let $\mathfrak{A}=F(a_1, \cdots, a_n)\subset\overline{F}_r$ be the algebra generated by the set (a_i) over F. One readily observes that since the matrices (a_i) generate $\overline{F}_r$, $\mathfrak{A}\otimes\overline{F}= \overline{F}(a_i)=\overline{F}_r$, where the product is taken over the centre of $\mathfrak{A}$. It follows, therefore, that $\mathfrak{A}$ is a central simple algebra of finite order over its centre, and hence A is primitive. Thus $F[x]/P\cong\mathfrak{A}$ means that P is a primitive ideal. Consequently, $P\supset J_k$. If $f(x)\in J_k$, then $f(x)\in P$, i.e., $f(x)$ is mapped onto zero by the preceding homomorphism which means that $f(a)=0$. This proves that $f(x)$ satisfies (Z_k^R).

The generality of Lemma 4 leads us to the following generalization of Theorem 1:

THEOREM 4. *The following conditions are equivalent*: (1) $f(x)\in J_k$; (2) $f(x)$ *satisfies* (Z_k^R); (3) $f(x)$ *generates a right* (*and left*) *nil ideal* mod Q_k.

Theorem 3 gives the equivalence of (1) and (2), and clearly (3) implies (1) since nil ideals are quasi-regular. The proof will be completed by showing that (2) implies (3). The proof of this fact is similar to the proof of Theorem 1: If $f(x)$ satisfies (Z_k^R) and H is any non-denumerable field-extension of $\overline{F}$, then, clearly, it follows by Lemma 4 that $f(x)$ vanishes also for all regular zeros of G in $\overline{H}_r$ with $r\leqq k$. Hence, $f(x)$ belongs to the Jacobson Radical of the quotient ring $H[x]/Q_H(G)$ (with the notations of the proof of Theorem 1). It follows by Lemma 1 that this radical is nil, thus $f(x)^m\in Q_H(G)$ for some integer m. Following the proof of Theorem 1 we obtain that we also have $f(x)^m \in Q(G)$. If $f(x)$ satisfies (Z_k^R) then obviously every polynomial of $f(x)F[x]$ satisfies the same condition. This implies that the right ideal $f(x)F[x]$ is nil modulo Q_k, which completes the proof of our theorem.

Theorem 4 states that the radical of $F[x]/Q_k$ is a nil ideal. This enables us to show:

THEOREM 5. *The Jacobson radical of a finitely generated algebra which satisfies an identity is a nil ideal.*[3]

Let $A = F(a_1, a_2, \cdots, a_n)$ be a finitely generated algebra over F satisfying a polynomial identity. Let $x_1, x_2, \cdots$ be an infinite set of noncommutative indeterminates over F and consider the free algebra $F[x_1, \cdots, x_i, \cdots]$. The correspondence: $x_i \rightarrow a_i$ for $i = 1, \cdots, n$ and $x_i \rightarrow 0$ for all the others x_i determines a homomorphism of $F[\cdots, x_i, \cdots]$ onto A. Let Q_∞ be the kernel of this homomorphism. Put $F[x] = F[x_1, \cdots, x_n]$ and $F[x_\infty] = F[x_1, x_2, \cdots]$. Let $Q = Q_\infty \cap F[x]$, then one readily observes that $F[x_\infty]/Q_\infty \cong F[x]/Q \cong A$. It follows by Lemma 1 of [1] that $Q_\infty \supset Q_0$ where Q_0 is a T-ideal in the sense of [1]. In view of Corollary 1 of [1] and of Theorem 6 of [1] it follows that the Jacobson radical of $F[x_\infty]/Q_0$ is an ideal M_k/Q_0 and the latter is nil. Clearly, $\mathfrak{M}_k = M_k \cap F[x]$.[4] We now obtain that $(M_k, Q_\infty)/Q_\infty$ is a nil ideal, since it is isomorphic with $M_k/(M_k \cap Q_\infty)$ and the latter is a homomorphic image of M_k/Q_0 because $M_k \cap Q_\infty \supset Q_0$. Let J_0/Q_∞ be the Jacobson radical of $F[x_\infty]/Q_\infty$ then since $(M_k, Q_\infty)/Q_\infty$ is nil it follows that $J_0 \supset (M_k, Q_\infty)$ and the Jacobson radical of $F[x_\infty]/(M_k, Q_\infty)$ is also $J_0/(M_k, Q_\infty)$. Consider now the isomorphism $F[x_\infty]/Q_\infty \cong F[x]/Q$ obtained by mapping $x_i \rightarrow x_i$ for $i = 1, \cdots, n$ and $x_i \rightarrow 0$ for all other indeterminates. This isomorphism maps $(M_k, Q_\infty)/Q_\infty$ onto $(\mathfrak{M}_k, Q)/Q$ and J_0/Q_∞ onto an ideal J/Q, and it induces, therefore, an isomorphism between $F[x_\infty]/(M_k, Q_\infty)$ and $F[x]/(\mathfrak{M}_k, Q)$. Consequently we obtain that $J/(\mathfrak{M}_k, Q)$ is the Jacobson radical of $F[x]/(\mathfrak{M}_k, Q)$. Now $(\mathfrak{M}_k, Q)$ satisfies the properties of the ideal Q_k which was used in the proofs of the preceding results; that is, it contains $\mathfrak{M}_k$. Hence by combining Theorems 3 and 4 it follows that the radical $J/(\mathfrak{M}_k, Q)$ is a nil ideal. By the previous results it follows that $(\mathfrak{M}_k, Q)/Q$ is nil. Hence J/Q is a nil ideal. Obviously, this ideal is an isomorphic image of the radical of A, hence the radical of the latter is also a nil ideal.

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[3] It is interesting to know if Theorem 5 is true for arbitrary finitely generated rings.

[4] Recall that M_k is the set of all polynomials $f(x) \in F[x_\infty]$ for which the relation $f(x) = 0$ holds identically in $\overline{F}_k$. The requirement that the identities are satisfied by $\overline{F}_k$ and not only by F_k is to make sure that the underlined field is infinite.

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HEBREW UNIVERSITY, JERUSALEM

A NOTE ON THE STONE-WEIERSTRASS THEOREM FOR QUATERNIONS[1]

JOHN C. HOLLADAY

A result of M. H. Stone [1, p. 466], which is nicely presented by N. Dunford [2, p. 23], is as follows: Let A be a closed subalgebra of the B-algebra $C(X)$ of all continuous real-valued functions on the compact Hausdorff space X. Then $A = C(X)$ if and only if A distinguishes between every pair of distinct points of X, i.e., for every pair $x_1 \neq x_2$ of points in X, there is an f in X such that $f(x_1) \neq f(x_2)$.

If one substitutes the word complex for the word real in the above statement, it becomes false. A well known counter example is obtained by letting X be the set of complex numbers z such that $|z| \leqq 1$ and letting A be the subalgebra of functions which are analytic in the interior of X.

The purpose of this note is to show that if the word quaternion is substituted for the word real in the above statement, it remains valid. To be specific, let A be a set of continuous quaternion-valued functions which satisfy the following conditions:

1. A is complete.
2. Given a quaternion q, the function $f(x) \equiv q$ is in A.
3. If f and g are in A, then fg and $f+g$ are in A.

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