ON THE ZEROES OF THE DERIVATIVES OF AN ENTIRE FUNCTION

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Let $f(z)$ be an entire function. We define the point set $L=L_f$ on the z-sphere as follows: $z_0 \in L$ *if and only if every neighborhood of* z_0 *contains zeros of infinitely many of the functions* $f^{(n)}(z)$. In [3] Pólya has named this set the *final set* of f, and collected various facts and conjectures about final sets.

The object of this paper is to prove that any compact subset of the sphere which contains the point ∞ is a set L_f. This result contrasts markedly with the theorem of Pólya [2] which completely characterizes the final sets of meromorphic functions; these sets are of a very special sort. In the same paper Pólya proved that *if* $f(z) = P(z)e^{Q(z)}$, *where* P *and* Q *are polynomials and* Q *is of degree* $q \geqq 2$, *then* L *consists of* q *equally-spaced rays emanating from one point.* This result has been extended by McLeod [1] to the case where $P(z)$ is a suitable canonical product.

All sets L_f have two properties: (1) L_f *is compact and* (2) *either* $\infty \in L_f$ *or* L_f *is void.* The first property is obvious. The second is a consequence of a theorem of Pólya and Saxer [4]: *if* $g(z)$ *is entire and not a polynomial then either* $g(z)=P(z) \exp Q(z)$ *where* P *and* Q *are polynomials, or* $gg'g''$ *has infinitely many zeros.* This result and the theorem of Pólya quoted earlier imply that $\infty \in L_f$ except for the cases (a) $f(z)=P(z)$, and (b) $f(z)=P(z) \exp(cz)$, where P is a polynomial and c a constant. Case (a) is trivial: L_f is the complete sphere. In case (b), if P is a constant then L_f is void. Otherwise set

$$f(z) = (az^m + bz^{m-1} + \cdots)e^{cz}, \qquad m > 0,\ a \neq 0.$$

Then

$$f^{(n)}(z) = (ac^n z^m + [bc + mna]c^{n-1}z^{m-1} + \cdots)e^{cz}$$

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and the arithmetic mean of the roots of $f^{(n)}$ is $-n/c+O(1)$; hence $\infty \in L_f$.

In the remainder of this paper we shall assume that $p(r)$ is defined for $r \geqq 0$ and satisfies

$$p(r) \in C, \qquad p(r) \uparrow \infty, \qquad p(0) > 0, \tag{1}$$

and

$$\lim_{r\to\infty} r^{-c}p(r) = \infty \qquad \textit{for every constant } c. \tag{2}$$

With these preliminaries we can now state our result.

THEOREM. *Let K be a compact subset of the complex sphere with $\infty \in K$, and let $p(r)$ satisfy* (1) *and* (2). *Then there exists an entire function f such that $L_f = K$ and such that the maximum modulus, $M(r)$, of f satisfies*

$$\limsup_{r\to\infty} \frac{M(r)}{p(r)} = 1. \tag{3}$$

We note that this theorem contains the answer to a question raised by Pólya [**3**, p. 181, note 2].

The proof requires the following lemma.

LEMMA. *Let s and ϵ be positive constants and let N and λ_0 be positive integers. Let a be any complex constant. Then there exists $c > 0$ and an integer $\lambda > \lambda_0$ such that the function*

$$g(z) = c(z-a)^\lambda \tag{4}$$

satisfies:

$$|g^{(n)}(z)| < \epsilon, \qquad 0 \leqq n \leqq N, \ |z| \leqq s, \tag{5}$$

$$|g(z)| \leqq p(|z|), \qquad |z| \geqq 0, \tag{6}$$

and there exists z_0, $|z_0| > s$, such that

$$|g(z_0)| = p(|z_0|). \tag{7}$$

PROOF OF LEMMA. We may assume that $a > 0$. Choose $r_0 > s+2a$. For each integer $\lambda > \lambda_0$, set $g_\lambda(z) = c_\lambda(z-a)^\lambda$, where $c_\lambda > 0$ is determined so that $g_\lambda(r_0) = p(r_0)$. For $|z| \leqq s$,

$$|g_\lambda(z)| \leqq c_\lambda(s+a)^\lambda = p(r_0)\left(\frac{s+a}{r_0-a}\right)^\lambda \to 0$$

as $\lambda \to \infty$. Hence if λ is sufficiently large, $g_\lambda(z)$ will satisfy (5) and also

$|g_\lambda(z)| < p(|z|)$ for $|z| \leqq s$. For such λ it follows from $g_\lambda(r_0) = p(r_0)$ and (2) that there exists c, $0 < c \leqq c_\lambda$ such that $g(z) = c(z-a)^\lambda$ satisfies (5), (6), and (7).

PROOF OF THEOREM. Let $\{a_n\}_1^\infty$ be a sequence of complex numbers such that the set of limits of all convergent subsequences of $\{a_n\}$ is exactly K. Set

$$f(z) = \sum_{n=1}^{\infty} c_n(z - a_n)^{\lambda_n} = \sum_{n=1}^{\infty} g_n(z), \tag{8}$$

where the c_n and λ_n are chosen as prescribed below. Let

$$A_n = \left\{ z \,\middle|\, |z - a_n| \geqq \frac{1}{n},\ |z| \leqq n + |a_n| \right\}, \tag{9}$$

and let C_n be the boundary of A_n. Let $c_1 = \lambda_1 = r_1 = \rho_1 = 1$. We choose the quadruple sequence $\{c_n, \lambda_n, \rho_n, r_n\}_1^\infty$ inductively. Having obtained $c_n, \lambda_n, \rho_n, r_n$, set

$$\inf_{0 \leqq m < \lambda_k} \left\{ \inf_{z \in A_k} \left| g_k^{(m)}(z) \right| \right\} = \delta_k > 0, \qquad k \leqq n, \tag{10}$$

and

$$\inf_{|z| \leqq k + |a_k|} \left| g_k^{(\lambda_k)}(z) \right| = \eta_k > 0, \qquad k \leqq n. \tag{11}$$

Then choose ρ_{n+1} such that

$$\rho_{n+1} \geqq \max_{1 \leqq k \leqq n} (k + |a_k|), \quad \rho_{n+1} > r_n,\ \rho_{n+1} > \rho_n, \tag{12}$$

and

$$\left| \sum_{k=1}^{n} g_k(z) \right| < \frac{p(r)}{n}, \qquad \text{for } |z| = r \geqq \rho_{n+1}. \tag{13}$$

By the lemma there exists $g_{n+1}(z)$ such that:

$$\lambda_{n+1} \geqq \lambda_n + 2, \tag{14}$$

$$\left| g_{n+1}^{(m)}(z) \right| < \frac{1}{2^{n+1}} \min_{1 \leqq k \leqq n} (\delta_k, \eta_k, 1), \quad \text{for } |z| \leqq \rho_{n+1},\ 0 \leqq m \leqq \lambda_n, \tag{15}$$

$$\left| g_{n+1}(z) \right| \leqq p(|z|), \qquad |z| \geqq 0, \tag{16}$$

and

$$\left| g_{n+1}(z_{n+1}) \right| = p(|z_{n+1}|), \tag{17}$$

for some z_{n+1} with

$$r_{n+1} = |z_{n+1}| > \rho_{n+1}. \tag{18}$$

It follows from (12) that $\rho_n \to \infty$, and the convergence of the series in (8) to an entire function $f(z)$ is a consequence of (15). Now

$$f^{(\lambda_m)}(z) = g_m^{(\lambda_m)}(z) + E_m(z) = \text{const.} + E_m(z)$$

and it follows from (15), (11), and (12) that

$$|E_m(z)| < \sum_{n=m+1}^{\infty} \frac{\eta_m}{2^n} = \frac{\eta_m}{2^m} < \eta_m \leqq |g_m^{(\lambda_m)}(z)|$$

for $|z| \leqq m + |a_m|$. Since $g_m^{(\lambda_m)}(z)$ has no zeros it follows from Rouché's theorem that

(19) *$f^{(\lambda_m)}(z)$ has no zeros in $|z| \leqq m + |a_m|$,* $\qquad m \geqq 1.$

If $\lambda_{m-1} < s < \lambda_m$, then $f^{(s)}(z) = g_m^{(s)}(z) + F_s(z)$, and it follows from (15), (10), and (12), that

$$|F_s(z)| < \sum_{n=m+1}^{\infty} \frac{\delta_m}{2^n} < \delta_m \leqq |g_m^{(s)}(z)|,$$

for $z \in C_m$; so by using Rouché's theorem twice we obtain:

(20) *$f^{(s)}(z)$ has one zero in $|z - a_m| < 1/m$ and no other zeros in $|z| \leqq m + |a_m|$.*

It is an obvious consequence of (19) and (20), together with the choice of the sequence $\{a_n\}$ that the final set of $f(z)$ is exactly K.

Now consider the growth of $f(z)$. For $\rho_n \leqq |z| = r \leqq \rho_{n+1}$,

$$|f(z)| \leqq \left|\sum_{k=1}^{n-1} g_k(z)\right| + |g_n(z)| + \left|\sum_{k=n+1}^{\infty} g_k(z)\right|.$$

By applying (13), (16), and (15) to these three terms we obtain

$$|f(z)| < \frac{p(r)}{n-1} + p(r) + 1, \qquad \rho_n \leqq |z| = r \leqq \rho_{n+1}.$$

Thus lim sup $(M(r)/p(r)) \leqq 1$. On the other hand, a similar argument, using (12), (17), and (18) shows that

$$M(r_n) \geqq -\frac{p(r_n)}{n-1} + p(r_n) - 1,$$

and hence lim sup $(M(r)/p(r)) \geqq 1$, which completes the proof of the theorem.

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AN ELEMENTARY PROOF OF THE CLOSURE IN L OF TRANSLATIONS OF e^{-x^2}, AND THE BOREL TAUBERIAN THEOREM

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It is well known that the Tauberian Theorem[1] for Borel summation can be deduced easily from the closure of translations of e^{-x^2} in $L(-\infty, \infty)$, this last result being a special case of Wiener's General Tauberian Theorem.[2]

A simple proof of the Littlewood Tauberian Theorem for Abel summation has been given by Karamata[3] by a method which depends on the fact that the closure theorem for the Abel kernel is closely related to the Weierstrass theorem on polynomial approximation to arbitrary functions and can be proved by elementary means. This suggests that it might be of interest to find elementary proofs of the closure theorems, and the associated Tauberian theorems, for other kernels by using their specific properties rather than Wiener's general theorem. We show here that this can be done very simply for the Borel kernel e^{-x^2}.

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