ON n-METACALORIC FUNCTIONS

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Introduction. Some theorems on polyharmonic functions can be generalized to n-metaharmonic functions, that is functions $u(x)$ ($x = (x_1, \cdots, x_N)$) satisfying equations of the form $p(\Delta)u(x) = 0$, where $p(t)$ is a polynomial in t of degree n and $\Delta = \sum \partial^2/\partial x_i^2$. We especially refer to [2] and [4].

In a recent paper, [5], Nicolesco has developed the theory of n-caloric functions, that is functions satisfying the parabolic equation $\Omega^n u(x, t) = 0$, where $\Omega u = \Delta u - \partial/\partial t$.

The aim of this paper is to generalize some of the theorems on n-caloric functions to n-metacaloric functions, that is to functions satisfying $p(\Omega)u(x, t) = 0$. Our results are analogous to the results of Ghermanesco [4] and the author [2], in the theory of $p(\Delta)u(x) = 0$. For the sake of simplicity we take $N = 1$, $\Delta = \partial^2/\partial x^2$, but all the results hold for the general case $\Delta = \sum \partial^2/\partial x_i^2$.

By a solution of $p(\Omega)u(x, t) = 0$, we shall always mean a function which possesses all the derivatives which appear in $p(\Omega)$ and which satisfies $p(\Omega)u(x, t) = 0$.

1. THEOREM 1. *Let $p(z)$ be the polynomial $\prod_{i=1}^k (z - \alpha_i)^{n_i}$ and consider the equation*

$$p(\Omega)u(x, t) = 0 \tag{1}$$

in a domain D. Then the general solution of (1) *has the form*

$$u(x, t) = \sum_{i=1}^{k} \sum_{j=0}^{n_i - 1} t^j u_j^{\alpha_i}(x, t), \tag{2}$$

where $u_j^{\alpha_i}(x, t)$ are solutions in D of $(\Omega - \alpha_i)u(x, t) = 0$.

PROOF. It can easily be proved by induction that if $v(x, t)$ satisfies $(\Omega - \alpha)v(x, t) = 0$, then

$$(\Omega - \alpha)^k (t^k v(x, t)) = (-1)^k k! v(x, t).$$

From this remark it follows that the u given by (2) satisfies (1). To prove the converse, consider first the case of $p(\Omega) = (\Omega - \alpha)^n$. The proof is by induction on n. It is sufficient to show that if u satisfies $(\Omega - \alpha)^n$

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$u=0$, then there exists a function $v(x, t)$ satisfying

$$(3) \qquad (\Omega - \alpha)v = 0, \qquad (\Omega - \alpha)^{n-1}(t^{n-1}v) = (\Omega - \alpha)^{n-1}u.$$

For then we write $u=t^{n-1}v+(u-t^{n-1}v)$ and use the inductive assumption. By the remark made at the beginning of the proof it is obvious that the function

$$v(x, t) = \frac{(-1)^{n-1}}{(n-1)!}(\Omega - \alpha)^{n-1}u(x, t)$$

satisfies (3).

Having proved the theorem for $p(\Omega)=(\Omega-\alpha)^n$, we can prove the general case of $p(\Omega)=\prod_{i=1}^{k}(\Omega-\alpha_i)^{n_i}$ by induction on k, in exactly the same way as in the case of $p(\Delta)$ (see [2]).

REMARK. That the representation (2) is unique, is easily seen by applying $\prod_{i=1;i\neq j}^{k}(\Omega-\alpha_i)^{n_i}(\Omega-\alpha_j)^m$ $m=n_j-1, \cdots, 1, 0$ to both sides of (2).

COROLLARY. *The solutions $u(x, t)$ of $p(\Omega)u=0$ have the following properties*:

(a) *$u(x, t)$ is infinitely differentiable in (x, t)*,

(b) *for every t_0, $u(x, t_0)$ is analytic in x, and*

(c) *for every x_0, $u(x_0, t)$ belongs to the second class of Holmgren* (see [3]).

The corollary follows from Theorem 1 and the fact that metacaloric functions possess the properties (a), (b), (c).

2. LEMMA 1. *Let $u(x, t)$ be a solution of $(\Omega-\alpha)u(x, t)=0$ in the strip $t_1\leqq t\leqq t_2$ and*

$$|u(x, t)| \leqq Me^{Kx^2} \qquad (t_1 \leqq t \leqq t_2).$$

Then for every $\epsilon>0$, there exists a constant $M'=M'(\epsilon, N, M)$ such that for t, $t_1+\epsilon\leqq t\leqq t_2$,

$$\left|\frac{\partial^k u(x, t)}{\partial x^{k-j}\partial t^j}\right| \leqq M'e^{2Kx^2} \qquad (0 \leqq j \leqq k, 0 \leqq k \leqq N).$$

PROOF. It is sufficient to prove that for every t_0, $t_1\leqq t_0<t_2$ and for every t which satisfies $t_0+\epsilon\leqq t\leqq t_0+1/8K-\epsilon$, $t\leqq t_2$,

$$\left|\frac{\partial^k u(x, t)}{\partial x^{k-j}\partial t^j}\right| \leqq M'e^{2Kx^2} \qquad (0 \leqq j \leqq k, 0 \leqq k \leqq N)$$

where M' depends on ϵ, N, M.

Without loss of generality, we may assume $t_0=0$. Consider the function

$$v(x, t) = \frac{e^{-\alpha t}}{2(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t} u(\xi, 0) d\xi$$

and apply to it the operator $\partial^k/\partial x^{k-j}\partial t^j$. Substituting $x-\xi=-2t^{1/2}s$ we have

$$\left| \frac{\partial^k v(x, t)}{\partial x^{k-j}\partial t^j} \right| \leqq \int_{-\infty}^{\infty} \left| \frac{\partial^k}{\partial x^{k-j}\partial t^j} \left(\frac{e^{-\alpha t} e^{-(x-\xi)^2/4t}}{2(\pi t)^{1/2}} \right) \right| M e^{K(x+2t^{1/2}s)^2} ds$$

$$\leqq M_1 \int_{-\infty}^{\infty} s^{2N} e^{-s^2} e^{2K(x^2+4ts^2)} ds \leqq M' e^{2Kx^2}.$$

It remains to prove that $v(x, t)=u(x, t)$. Since the function $w=u-v$ satisfies

$$w(x, 0) = 0, \qquad (\Omega - \alpha) w(x, t) = 0, \qquad | w(x, t) | \leqq (M + M') e^{2Kx^2},$$

it follows from [1] that $w(x, t) \equiv 0$.

3. THEOREM 2. *Let $u(x, t)$ be a solution of $(\Omega-\alpha)u=0$ in the strip $t_1 \leqq t \leqq t_2$ and $|u(x, t)| \leqq Me^{Kx^2}$ $(t_1 \leqq t \leqq t_2)$. Then for every h, t which satisfy $t_1 < t \leqq t_2$, $t_1 < t-h$, $0<h<1/8K$, the following mean-value formula holds*:

$$\mu(u; h) \equiv \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} u(\xi, t - h) d\xi = e^{\alpha h} u(x, t). \tag{5}$$

PROOF. Let $\phi(h)=2(\pi h)^{1/2}\mu(u; h)$. Using Lemma 1 and integrating by parts, we have

$$\frac{\partial \phi}{\partial h} = \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} \left[\Omega(u\xi, t - h) + \frac{1}{2h} u(\xi, t - h) \right] d\xi = \left(\alpha + \frac{1}{2h} \right) \phi,$$

or $\phi(h)=ch^{1/2}e^{\alpha h}$. To find c, observe that $\mu(u; h)=u(x, t)+o(1)$ $(h \to 0)$, so that $c=2(\pi)^{1/2}u(x, t)$ and we have proved (5).

4. DEFINITION. A function $u(x, t)$ defined in a strip $t_1 \leqq t \leqq t_2$, is said to be of type (K, n) (we suppose $K \neq 0$), if $\Omega^k u(x, t)$ $(k=0, 1, \cdots, n)$ exist and satisfy

$$| \Omega^k u(x, t) | \leqq M e^{Kx^2} \qquad (t_1 \leqq t \leqq t_2,\ k = 0, 1, \cdots, n),$$

and of type $(0, n)$, if for every positive ϵ it is of type (ϵ, n).

THEOREM 3. *Let $u(x, t)$ be a solution of $p(\Omega)u=0$, where $p(z)$*

$=\prod_{i=1}^{k}(z-\alpha_i)^{n_i}$ $(\sum n_i=n)$, *and suppose* u *to be of type* $(K, n-1)$ *in the strip* $t_1\leqq t\leqq t_2$. *Then, for every* t, h *which satisfy*

$$(6)\qquad \begin{gathered} t_1 < t \leqq t_2, \qquad t_1 < t-h, \qquad 0 < h < 1/8K, \\ \mu(u;\, h) = \sum_{i=1}^{k}\sum_{j=0}^{n_i-1}(t-h)^j e^{\alpha_i h} u_j^{\alpha_i}(x,\, t). \end{gathered}$$

PROOF. Since $u(x, t)$ is of type $(K, n-1)$, for every polynomial $q(z)$ of degree $\leqq n-1$, $|q(\Omega)u(x,t)| \leqq Me^{Kx^2}$. Choosing $q(z)=\prod_{i=1;i\neq j}^{k}(z-\alpha_i)^{n_i}(z-\alpha_j)^m$, $m=n_j-1, n_j-2, \cdots, 0$ we can easily see that each $u_j^{\alpha t}(x, t)$ in (2) is of type $(K, 0)$. Therefore, by Theorem 2

$$\begin{aligned} \mu(u;\, h) &= \sum_{i=1}^{k}\sum_{j=0}^{n_i-1}\frac{1}{2(\pi h)^{1/2}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h}(t-h)^j u_j^{\alpha_i}(\xi,\, t-h)d\xi \\ &= \sum_{i=1}^{k}\sum_{j=0}^{n_i-1}(t-h)^j \mu(u_j^{\alpha_i};\, h) = \sum_{i=1}^{k}\sum_{j=0}^{n_i-1}(t-h)^j e^{\alpha_i h} u_j^{\alpha_i}(x,\, t). \end{aligned}$$

5. We now prove a generalization of Liouville's Theorem:

THEOREM 4. *Let* $p(z)$ *be a polynomial of the form* $\prod_{i=1}^{k}(z-\alpha_i)^{n_i}$ $(\sum n_i=n)$ *with* $\operatorname{Re}\{\alpha_i\}\geqq 0$, *and let* $u(x, t)$ *satisfy* $p(\Omega)u(x, t)=0$ *in the strip* $-\infty<t\leqq t_2$. *Suppose* $u(x, t)$ *to be of the type* $(0, n-1)$ *in every strip* $t_1\leqq t\leqq t_2$. *If* $u(x, t)$ *is bounded and if*

$$(7)\qquad \lim_{h\to\infty}\mu(u;\, h)\ \textit{exists},$$

then $u(x, t)\equiv const.$

PROOF. Using Theorem 1 with D: $-\infty<t\leqq t_2$, we derive equation (6) for all positive h. Write (6) in the form

$$(8)\qquad \mu(u;\, h) = \sum_{i=1}^{n} F_i(x,\, t)\phi_i(h).$$

Letting $h\to\infty$ and comparing the behaviour at infinity of the functions $\phi_i(h)$, we conclude that

$$(9)\qquad \mu(u;\, h) = \sum_{\operatorname{Re}\{\alpha_i\}=0;\,\alpha_i\neq 0} e^{\alpha_i h} u_0^{\alpha_i}(x,\, t) + u_0^{0}(x,\, t).$$

The sum on the right-hand side in (9) is an almost periodic function in the sense of Bohr, and since its limit (as $h\to\infty$) exists, it must vanish. We obtain

$$\mu(u;\, h) = u_0^{0}(x,\, t).$$

Noting that $\mu(u;h)=u(x,t)+o(1)$ $(h\to 0)$, we conclude

$$\mu(u;h) = u(x,t). \tag{10}$$

By [5], $\mu(u;h)=u(x,t)+h\Omega u(x,t)+o(h)$ $(h\to 0)$ and together with (10), $\Omega u(x,t)=0$. Since for caloric functions Liouville's Theorem is true (see [5]), the proof is completed.

REMARK. From the proof it follows that in case that either Re $\{\alpha_i\}>0$ or $\alpha_i=0$ $(1\leqq i\leqq k)$, the assumption (7) is superfluous.

6. In this section we prove that equation (8) characterizes the n-metacaloric functions.

THEOREM 5. *Let $u(x,t)$ be defined in the strip $t_1\leqq t\leqq t_2$ and possess second-order continuous derivatives which satisfy*

$$|u(x,t)|, \quad \left|\frac{\partial u(x,t)}{\partial t}\right|, \quad \left|\frac{\partial u(x,t)}{\partial x}\right|, \quad \left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \leqq Me^{Kx^2}.$$

Then for every t, h which satisfy

$$\begin{gathered} t\leqq t_2, \qquad t_1\leqq t-h<t_2 \qquad 0<h<\frac{1}{4K}, \\ \Omega\mu(u;h) = \frac{\partial \mu(u;h)}{\partial h}. \end{gathered} \tag{11}$$

PROOF. As in the proof of Theorem 3,

$$\frac{\partial(2(\pi h)^{1/2}\mu)}{\partial h} = \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h}\left(\Omega u + \frac{1}{2h}u\right)d\xi.$$

Since on the other hand

$$\frac{\partial(2(\pi h)^{1/2}\mu)}{\partial h} = 2(\pi h)^{1/2}\frac{\partial\mu}{\partial h} + \frac{2\pi^{1/2}}{2h^{1/2}}\mu,$$

we conclude

$$\frac{\partial\mu}{\partial h} = \frac{1}{2(\pi h)^{1/2}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h}\Omega u(\xi, t-h)d\xi = \Omega\mu.$$

Suppose that each $\Omega^k u(x,t)$ $(k=0,1,\cdots,n-1)$ satisfies the assumptions of Theorem 5. We then have

$$\begin{aligned} \Omega^k\mu(u;h) &= \frac{1}{2(\pi h)^{1/2}}\int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h}\Omega^k u(\xi, t-h)d\xi \\ &= \Omega^k u(x,t) + o(1) \qquad (k=0,1,\cdots,n;\ h\to 0). \end{aligned} \tag{12}$$

Suppose $u(x, t)$ satisfies (8) with n times differentiable $\phi_i(h)$. Then by Theorem 5

$$\Omega^k\mu = \sum_{i=1}^{n} F_i(x, t) \frac{d^k\phi_i(h)}{dh^k} \qquad (k = 0, 1, \cdots, n).$$

Since this system has a nontrivial solution, we conclude that

$$(13) \qquad \begin{vmatrix} \mu(u; h) & \phi_1(h) & \cdots \phi_n(h) \\ \Omega\mu(u; h) & \phi_1'(h) & \cdots \phi_n'(h) \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ \Omega^n\mu(u; h) & \phi_1^{(n)}(h) & \cdots \phi_n^{(n)}(h) \end{vmatrix} = 0.$$

Taking $h \to 0$ and using (12) it follows that u is n-metacaloric. Hence

THEOREM 6. *If $u(x, t)$ is n-metacaloric and of type $(K, n-1)$ in the strip $t_1 \leqq t \leqq t_2$, then*

$$\mu(u; h) = \sum_{i=1}^{n} F_i(x, t)\phi_i(h) \qquad (t_1 < t - h, t < t_2, 0 < h < 1/8K),$$

$\phi_i(h)$ are analytic in h and $F_i(x, t)$ are indefinitely differentiable in (x, t) and analytic in x. Conversely, if

$$(14) \qquad \begin{gathered} \left| \Omega^k u(x, t) \right|, \quad \left| \frac{\partial}{\partial t} \Omega^k u(x, t) \right|, \quad \left| \frac{\partial}{\partial x} \Omega^k u(x, t) \right|, \\ \left| \frac{\partial^2}{\partial x^2} \Omega^k u(x, t) \right| \leqq M e^{Kx^2} \quad (k = 0, \cdots, n-1) \end{gathered}$$

and if (8) *holds with n times differentiable $\phi_i(h)$ and for h sufficiently small, then $u(x, t)$ is n-metacaloric and of type $(K, n-1)$.*

7. LEMMA 2. *Let $u(x, t)$ be n-metacaloric and of type $(K, n-1)$ in the strip $t_1 \leqq t \leqq t_2$. Then* (14) *holds in any interval $t_1 + \epsilon \leqq t \leqq t_2$ and with $2K$ in place of K.*

PROOF. Write u in the form (2). As in the proof of Theorem 3, each $u_j^{\alpha_i}$ is of type $(K, 0)$. Apply Lemma 1 to $u_j^{\alpha_i}$.

Suppose $u(x, t)$ to be n-metacaloric and of type $(K, n-1)$. Using Lemma 2, we have

$$\Omega^k\mu(u; h) = \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} \Omega^k u(\xi, t - h) d\xi.$$

Since $p(\Omega)u = 0$, we also have $p(\Omega)\mu(u; h) = 0$. By Theorem 5 we get

$$(15) \qquad p\left(\frac{\partial}{\partial h}\right)\mu(u; h) = 0.$$

Conversely, if $\mu(u;\, h)$ satisfies (15) and u satisfies (14), then by Theorem 5 and (15) we get $p(\Omega)\mu(u;\, h)=0$. Taking $h\rightarrow 0$, we conclude that $p(\Omega)u(x,\, t)=0$. We have proved:

THEOREM 7. *If $u(x,t)$ satisfies $p(\Omega)u=0$ and if it is of type $(K, n-1)$, then*

$$p(\partial/\partial h)\mu(u;\, h) = 0 \qquad (t_1 < t - h < t_2,\ 0 < h < 1/8K). \tag{16}$$

Conversely, from (14) *and* (16) *follows* $p(\Omega)u=0$.

8. We shall derive a special form of equation (8). From (6) it is clear that the $\phi_i(h)$ are independent solutions of $p(\partial/\partial h)\phi(h)=0$, therefore their Wronskian is different from zero. Applying Ω^k to both sides of (8) and letting $h\rightarrow 0$, we get the system

$$\mu(u;\, h) = \sum_{i=1}^{n} F_i(x,\, t)\phi_i(h),$$

$$\Omega^k u(x,\, t) = \sum_{i=1}^{n} F_i(x,\, t)\frac{d^k\phi_i(0)}{dh^k} \qquad (k = 0, 1, \cdots, n-1),$$

from which it follows that

$$\begin{vmatrix} \mu(u;\, h) & \phi_1(h) & \cdots & \phi_n(h) \\ u(x,\, t) & \phi_1(0) & \cdots & \phi_n(0) \\ \cdot & \cdot & \cdot & \cdot \\ \Omega^{n-1}u(x,t) & \phi_1^{(n-1)}(0) & \cdots & \phi_n^{(n-1)}(0) \end{vmatrix} = 0,$$

or

$$\mu(u;\, h) = \sum_{k=0}^{n-1} c_k(h)\Omega^k u(x,\, t),$$

where $c_k(h)$ are linear combinations of expressions $h^j e^{\alpha_i h}$.

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