

ON n -METACALORIC FUNCTIONS

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Introduction. Some theorems on polyharmonic functions can be generalized to n -metaharmonic functions, that is functions $u(x)$ ($x = (x_1, \dots, x_N)$) satisfying equations of the form $p(\Delta)u(x) = 0$, where $p(t)$ is a polynomial in t of degree n and $\Delta = \sum \partial^2 / \partial x_i^2$. We especially refer to [2] and [4].

In a recent paper, [5], Nicolesco has developed the theory of n -caloric functions, that is functions satisfying the parabolic equation $\Omega^n u(x, t) = 0$, where $\Omega u = \Delta u - \partial / \partial t$.

The aim of this paper is to generalize some of the theorems on n -caloric functions to n -metacaloric functions, that is to functions satisfying $p(\Omega)u(x, t) = 0$. Our results are analogous to the results of Ghermanesco [4] and the author [2], in the theory of $p(\Delta)u(x) = 0$. For the sake of simplicity we take $N = 1$, $\Delta = \partial^2 / \partial x^2$, but all the results hold for the general case $\Delta = \sum \partial^2 / \partial x_i^2$.

By a solution of $p(\Omega)u(x, t) = 0$, we shall always mean a function which possesses all the derivatives which appear in $p(\Omega)$ and which satisfies $p(\Omega)u(x, t) = 0$.

1. **THEOREM 1.** *Let $p(z)$ be the polynomial $\prod_{i=1}^k (z - \alpha_i)^{n_i}$ and consider the equation*

$$(1) \quad p(\Omega)u(x, t) = 0$$

in a domain D . Then the general solution of (1) has the form

$$(2) \quad u(x, t) = \sum_{i=1}^k \sum_{j=0}^{n_i-1} t^j u_j^{\alpha_i}(x, t),$$

where $u_j^{\alpha_i}(x, t)$ are solutions in D of $(\Omega - \alpha_i)u(x, t) = 0$.

PROOF. It can easily be proved by induction that if $v(x, t)$ satisfies $(\Omega - \alpha)v(x, t) = 0$, then

$$(\Omega - \alpha)^k (t^k v(x, t)) = (-1)^k k! v(x, t).$$

From this remark it follows that the u given by (2) satisfies (1). To prove the converse, consider first the case of $p(\Omega) = (\Omega - \alpha)^n$. The proof is by induction on n . It is sufficient to show that if u satisfies $(\Omega - \alpha)^n$

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$u = 0$, then there exists a function $v(x, t)$ satisfying

$$(3) \quad (\Omega - \alpha)v = 0, \quad (\Omega - \alpha)^{n-1}(t^{n-1}v) = (\Omega - \alpha)^{n-1}u.$$

For then we write $u = t^{n-1}v + (u - t^{n-1}v)$ and use the inductive assumption. By the remark made at the beginning of the proof it is obvious that the function

$$v(x, t) = \frac{(-1)^{n-1}}{(n-1)!} (\Omega - \alpha)^{n-1}u(x, t)$$

satisfies (3).

Having proved the theorem for $p(\Omega) = (\Omega - \alpha)^n$, we can prove the general case of $p(\Omega) = \prod_{i=1}^k (\Omega - \alpha_i)^{n_i}$ by induction on k , in exactly the same way as in the case of $p(\Delta)$ (see [2]).

REMARK. That the representation (2) is unique, is easily seen by applying $\prod_{i=1, i \neq j}^k (\Omega - \alpha_i)^{n_i} (\Omega - \alpha_j)^m$ $m = n_j - 1, \dots, 1, 0$ to both sides of (2).

COROLLARY. *The solutions $u(x, t)$ of $p(\Omega)u = 0$ have the following properties:*

- (a) $u(x, t)$ is infinitely differentiable in (x, t) ,
- (b) for every t_0 , $u(x, t_0)$ is analytic in x , and
- (c) for every x_0 , $u(x_0, t)$ belongs to the second class of Holmgren (see [3]).

The corollary follows from Theorem 1 and the fact that metacaloric functions possess the properties (a), (b), (c).

2. LEMMA 1. *Let $u(x, t)$ be a solution of $(\Omega - \alpha)u(x, t) = 0$ in the strip $t_1 \leq t \leq t_2$ and*

$$|u(x, t)| \leq M e^{Kx^2} \quad (t_1 \leq t \leq t_2).$$

Then for every $\epsilon > 0$, there exists a constant $M' = M'(\epsilon, N, M)$ such that for t , $t_1 + \epsilon \leq t \leq t_2$,

$$\left| \frac{\partial^k u(x, t)}{\partial x^{k-i} \partial t^i} \right| \leq M' e^{2Kx^2} \quad (0 \leq j \leq k, 0 \leq k \leq N).$$

PROOF. It is sufficient to prove that for every t_0 , $t_1 \leq t_0 < t_2$ and for every t which satisfies $t_0 + \epsilon \leq t \leq t_0 + 1/8K - \epsilon$, $t \leq t_2$,

$$\left| \frac{\partial^k u(x, t)}{\partial x^{k-i} \partial t^i} \right| \leq M' e^{2Kx^2} \quad (0 \leq j \leq k, 0 \leq k \leq N)$$

where M' depends on ϵ, N, M .

Without loss of generality, we may assume $t_0=0$. Consider the function

$$v(x, t) = \frac{e^{-\alpha t}}{2(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t} u(\xi, 0) d\xi$$

and apply to it the operator $\partial^k/\partial x^{k-i}\partial t^i$. Substituting $x-\xi = -2t^{1/2}s$ we have

$$\begin{aligned} \left| \frac{\partial^k v(x, t)}{\partial x^{k-i}\partial t^i} \right| &\leq \int_{-\infty}^{\infty} \left| \frac{\partial^k}{\partial x^{k-i}\partial t^i} \left(\frac{e^{-\alpha t} e^{-(x-\xi)^2/4t}}{2(\pi t)^{1/2}} \right) \right| M e^{K(x+2t^{1/2}s)^2} ds \\ &\leq M_1 \int_{-\infty}^{\infty} s^{2N} e^{-s^2} e^{2K(x^2+4ts^2)} ds \leq M' e^{2Kx^2}. \end{aligned}$$

It remains to prove that $v(x, t) = u(x, t)$. Since the function $w = u - v$ satisfies

$$w(x, 0) = 0, \quad (\Omega - \alpha)w(x, t) = 0, \quad |w(x, t)| \leq (M + M')e^{2Kx^2},$$

it follows from [1] that $w(x, t) \equiv 0$.

3. THEOREM 2. Let $u(x, t)$ be a solution of $(\Omega - \alpha)u = 0$ in the strip $t_1 \leq t \leq t_2$ and $|u(x, t)| \leq M e^{Kx^2}$ ($t_1 \leq t \leq t_2$). Then for every h, t which satisfy $t_1 < t \leq t_2, t_1 < t - h, 0 < h < 1/8K$, the following mean-value formula holds:

$$(5) \quad \mu(u; h) \equiv \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} u(\xi, t - h) d\xi = e^{\alpha h} u(x, t).$$

PROOF. Let $\phi(h) = 2(\pi h)^{1/2} \mu(u; h)$. Using Lemma 1 and integrating by parts, we have

$$\frac{\partial \phi}{\partial h} = \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} \left[\Omega(u\xi, t - h) + \frac{1}{2h} u(\xi, t - h) \right] d\xi = \left(\alpha + \frac{1}{2h} \right) \phi,$$

or $\phi(h) = ch^{1/2} e^{\alpha h}$. To find c , observe that $\mu(u; h) = u(x, t) + o(1)$ ($h \rightarrow 0$), so that $c = 2(\pi)^{1/2} u(x, t)$ and we have proved (5).

4. DEFINITION. A function $u(x, t)$ defined in a strip $t_1 \leq t \leq t_2$, is said to be of type (K, n) (we suppose $K \neq 0$), if $\Omega^k u(x, t)$ ($k = 0, 1, \dots, n$) exist and satisfy

$$|\Omega^k u(x, t)| \leq M e^{Kx^2} \quad (t_1 \leq t \leq t_2, k = 0, 1, \dots, n),$$

and of type $(0, n)$, if for every positive ϵ it is of type (ϵ, n) .

THEOREM 3. Let $u(x, t)$ be a solution of $p(\Omega)u = 0$, where $p(z)$

$= \prod_{i=1}^k (z - \alpha_i)^{n_i}$ ($\sum n_i = n$), and suppose u to be of type $(K, n - 1)$ in the strip $t_1 \leq t \leq t_2$. Then, for every t, h which satisfy

$$(6) \quad t_1 < t \leq t_2, \quad t_1 < t - h, \quad 0 < h < 1/8K,$$

$$\mu(u; h) = \sum_{i=1}^k \sum_{j=0}^{n_i-1} (t - h)^j e^{\alpha_i h} u_j^{\alpha_i}(x, t).$$

PROOF. Since $u(x, t)$ is of type $(K, n - 1)$, for every polynomial $q(z)$ of degree $\leq n - 1$, $|q(\Omega)u(x, t)| \leq Me^{Kx^2}$. Choosing $q(z) = \prod_{i=1}^k (z - \alpha_i)^{n_i} (z - \alpha_j)^m$, $m = n_j - 1, n_j - 2, \dots, 0$ we can easily see that each $u_j^{\alpha_i}(x, t)$ in (2) is of type $(K, 0)$. Therefore, by Theorem 2

$$\begin{aligned} \mu(u; h) &= \sum_{i=1}^k \sum_{j=0}^{n_i-1} \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} (t - h)^j u_j^{\alpha_i}(\xi, t - h) d\xi \\ &= \sum_{i=1}^k \sum_{j=0}^{n_i-1} (t - h)^j \mu(u_j^{\alpha_i}; h) = \sum_{i=1}^k \sum_{j=0}^{n_i-1} (t - h)^j e^{\alpha_i h} u_j^{\alpha_i}(x, t). \end{aligned}$$

5. We now prove a generalization of Liouville's Theorem:

THEOREM 4. Let $p(z)$ be a polynomial of the form $\prod_{i=1}^k (z - \alpha_i)^{n_i}$ ($\sum n_i = n$) with $\text{Re} \{ \alpha_i \} \geq 0$, and let $u(x, t)$ satisfy $p(\Omega)u(x, t) = 0$ in the strip $-\infty < t \leq t_2$. Suppose $u(x, t)$ to be of the type $(0, n - 1)$ in every strip $t_1 \leq t \leq t_2$. If $u(x, t)$ is bounded and if

$$(7) \quad \lim_{h \rightarrow \infty} \mu(u; h) \text{ exists,}$$

then $u(x, t) \equiv \text{const.}$

PROOF. Using Theorem 1 with $D: -\infty < t \leq t_2$, we derive equation (6) for all positive h . Write (6) in the form

$$(8) \quad \mu(u; h) = \sum_{i=1}^n F_i(x, t) \phi_i(h).$$

Letting $h \rightarrow \infty$ and comparing the behaviour at infinity of the functions $\phi_i(h)$, we conclude that

$$(9) \quad \mu(u; h) = \sum_{\text{Re}\{\alpha_i\}=0; \alpha_i \neq 0} e^{\alpha_i h} u_0^{\alpha_i}(x, t) + u_0^0(x, t).$$

The sum on the right-hand side in (9) is an almost periodic function in the sense of Bohr, and since its limit (as $h \rightarrow \infty$) exists, it must vanish. We obtain

$$\mu(u; h) = u_0^0(x, t).$$

Noting that $\mu(u; h) = u(x, t) + o(1)$ ($h \rightarrow 0$), we conclude

$$(10) \quad \mu(u; h) = u(x, t).$$

By [5], $\mu(u; h) = u(x, t) + h\Omega u(x, t) + o(h)$ ($h \rightarrow 0$) and together with (10), $\Omega u(x, t) = 0$. Since for caloric functions Liouville's Theorem is true (see [5]), the proof is completed.

REMARK. From the proof it follows that in case that either $\text{Re} \{ \alpha_i \} > 0$ or $\alpha_i = 0$ ($1 \leq i \leq k$), the assumption (7) is superfluous.

6. In this section we prove that equation (8) characterizes the n -metacaloric functions.

THEOREM 5. Let $u(x, t)$ be defined in the strip $t_1 \leq t \leq t_2$ and possess second-order continuous derivatives which satisfy

$$\left| u(x, t) \right|, \quad \left| \frac{\partial u(x, t)}{\partial t} \right|, \quad \left| \frac{\partial u(x, t)}{\partial x} \right|, \quad \left| \frac{\partial^2 u(x, t)}{\partial x^2} \right| \leq M e^{Kx^2}.$$

Then for every t, h which satisfy

$$(11) \quad t \leq t_2, \quad t_1 \leq t - h < t_2 \quad 0 < h < \frac{1}{4K},$$

$$\Omega \mu(u; h) = \frac{\partial \mu(u; h)}{\partial h}.$$

PROOF. As in the proof of Theorem 3,

$$\frac{\partial(2(\pi h)^{1/2} \mu)}{\partial h} = \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} \left(\Omega u + \frac{1}{2h} u \right) d\xi.$$

Since on the other hand

$$\frac{\partial(2(\pi h)^{1/2} \mu)}{\partial h} = 2(\pi h)^{1/2} \frac{\partial \mu}{\partial h} + \frac{2\pi^{1/2}}{2h^{1/2}} \mu,$$

we conclude

$$\frac{\partial \mu}{\partial h} = \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} \Omega u(\xi, t - h) d\xi = \Omega \mu.$$

Suppose that each $\Omega^k u(x, t)$ ($k = 0, 1, \dots, n - 1$) satisfies the assumptions of Theorem 5. We then have

$$(12) \quad \begin{aligned} \Omega^k \mu(u; h) &= \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} \Omega^k u(\xi, t - h) d\xi \\ &= \Omega^k u(x, t) + o(1) \quad (k = 0, 1, \dots, n; h \rightarrow 0). \end{aligned}$$

Suppose $u(x, t)$ satisfies (8) with n times differentiable $\phi_i(h)$. Then by Theorem 5

$$\Omega^k \mu = \sum_{i=1}^n F_i(x, t) \frac{d^k \phi_i(h)}{dh^k} \quad (k = 0, 1, \dots, n).$$

Since this system has a nontrivial solution, we conclude that

$$(13) \quad \begin{vmatrix} \mu(u; h) & \phi_1(h) & \dots & \phi_n(h) \\ \Omega \mu(u; h) & \phi_1'(h) & \dots & \phi_n'(h) \\ \dots & \dots & \dots & \dots \\ \Omega^n \mu(u; h) & \phi_1^{(n)}(h) & \dots & \phi_n^{(n)}(h) \end{vmatrix} = 0.$$

Taking $h \rightarrow 0$ and using (12) it follows that u is n -metacaloric. Hence

THEOREM 6. *If $u(x, t)$ is n -metacaloric and of type $(K, n-1)$ in the strip $t_1 \leq t \leq t_2$, then*

$$\mu(u; h) = \sum_{i=1}^n F_i(x, t) \phi_i(h) \quad (t_1 < t - h, t < t_2, 0 < h < 1/8K),$$

$\phi_i(h)$ are analytic in h and $F_i(x, t)$ are indefinitely differentiable in (x, t) and analytic in x . Conversely, if

$$(14) \quad \left| \Omega^k u(x, t) \right|, \quad \left| \frac{\partial}{\partial t} \Omega^k u(x, t) \right|, \quad \left| \frac{\partial}{\partial x} \Omega^k u(x, t) \right|, \\ \left| \frac{\partial^2}{\partial x^2} \Omega^k u(x, t) \right| \leq M e^{Kx^2} \quad (k = 0, \dots, n-1)$$

and if (8) holds with n times differentiable $\phi_i(h)$ and for h sufficiently small, then $u(x, t)$ is n -metacaloric and of type $(K, n-1)$.

7. LEMMA 2. *Let $u(x, t)$ be n -metacaloric and of type $(K, n-1)$ in the strip $t_1 \leq t \leq t_2$. Then (14) holds in any interval $t_1 + \epsilon \leq t \leq t_2$ and with $2K$ in place of K .*

PROOF. Write u in the form (2). As in the proof of Theorem 3, each $u_j^{a_i}$ is of type $(K, 0)$. Apply Lemma 1 to $u_j^{a_i}$.

Suppose $u(x, t)$ to be n -metacaloric and of type $(K, n-1)$. Using Lemma 2, we have

$$\Omega^k \mu(u; h) = \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} \Omega^k u(\xi, t-h) d\xi.$$

Since $p(\Omega)u=0$, we also have $p(\Omega)\mu(u; h)=0$. By Theorem 5 we get

$$(15) \quad p\left(\frac{\partial}{\partial h}\right)\mu(u; h) = 0.$$

Conversely, if $\mu(u; h)$ satisfies (15) and u satisfies (14), then by Theorem 5 and (15) we get $p(\Omega)\mu(u; h) = 0$. Taking $h \rightarrow 0$, we conclude that $p(\Omega)u(x, t) = 0$. We have proved:

THEOREM 7. *If $u(x, t)$ satisfies $p(\Omega)u = 0$ and if it is of type $(K, n - 1)$, then*

$$(16) \quad p(\partial/\partial h)\mu(u; h) = 0 \quad (t_1 < t - h < t_2, 0 < h < 1/8K).$$

Conversely, from (14) and (16) follows $p(\Omega)u = 0$.

8. We shall derive a special form of equation (8). From (6) it is clear that the $\phi_i(h)$ are independent solutions of $p(\partial/\partial h)\phi(h) = 0$, therefore their Wronskian is different from zero. Applying Ω^k to both sides of (8) and letting $h \rightarrow 0$, we get the system

$$\begin{aligned} \mu(u; h) &= \sum_{i=1}^n F_i(x, t)\phi_i(h), \\ \Omega^k u(x, t) &= \sum_{i=1}^n F_i(x, t) \frac{d^k \phi_i(0)}{dh^k} \quad (k = 0, 1, \dots, n - 1), \end{aligned}$$

from which it follows that

$$\begin{vmatrix} \mu(u; h) & \phi_1(h) & \cdots & \phi_n(h) \\ u(x, t) & \phi_1(0) & \cdots & \phi_n(0) \\ \dots & \dots & \dots & \dots \\ \Omega^{n-1} u(x, t) & \phi_1^{(n-1)}(0) & \cdots & \phi_n^{(n-1)}(0) \end{vmatrix} = 0,$$

or

$$\mu(u; h) = \sum_{k=0}^{n-1} c_k(h)\Omega^k u(x, t),$$

where $c_k(h)$ are linear combinations of expressions $h^i e^{\alpha_i h}$.

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