ON \( n \)-METACALORIC FUNCTIONS

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Introduction. Some theorems on polyharmonic functions can be generalized to \( n \)-metaharmonic functions, that is functions \( u(x) \) \((x = (x_1, \cdots, x_N))\) satisfying equations of the form \( p(\Delta)u(x) = 0 \), where \( p(t) \) is a polynomial in \( t \) of degree \( n \) and \( \Delta = \sum \partial^2/\partial x_i^2 \). We especially refer to [2] and [4].

In a recent paper, [5], Nicolesco has developed the theory of \( n \)-caloric functions, that is functions satisfying the parabolic equation \( \Omega^nu(x, t) = 0 \), where \( \Omega u = \Delta u - \partial/\partial t \).

The aim of this paper is to generalize some of the theorems on \( n \)-caloric functions to \( n \)-metacaloric functions, that is to functions satisfying \( p(\Omega)u(x, t) = 0 \). Our results are analogous to the results of Ghermanesco [4] and the author [2], in the theory of \( p(\Delta)u(x) = 0 \). For the sake of simplicity we take \( N = 1, \Delta = \partial^2/\partial x^2 \), but all the results hold for the general case \( \Delta = \sum \partial^2/\partial x_i^2 \).

By a solution of \( p(\Omega)u(x, t) = 0 \), we shall always mean a function which possesses all the derivatives which appear in \( p(\Omega) \) and which satisfies \( p(\Omega)u(x, t) = 0 \).

1. Theorem 1. Let \( p(z) \) be the polynomial \( \prod_{i=1}^k (z - \alpha_i)^{n_i} \) and consider the equation

\[
p(\Omega)u(x, t) = 0
\]

in a domain \( D \). Then the general solution of (1) has the form

\[
u(x, t) = \sum_{i=1}^k \sum_{j=0}^{n_i-1} t^j u_i^j(x, t),
\]

where \( u_i^j(x, t) \) are solutions in \( D \) of \((\Omega - \alpha_i)u(x, t) = 0\).

Proof. It can easily be proved by induction that if \( v(x, t) \) satisfies \((\Omega - \alpha)v(x, t) = 0\), then

\[
(\Omega - \alpha)^k(t^k v(x, t)) = (-1)^k k! v(x, t).
\]

From this remark it follows that the \( u \) given by (2) satisfies (1). To prove the converse, consider first the case of \( p(\Omega) = (\Omega - \alpha)^n \). The proof is by induction on \( n \). It is sufficient to show that if \( u \) satisfies \((\Omega - \alpha)^n \)

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u = 0, then there exists a function \( v(x, t) \) satisfying

\[
(\Omega - \alpha)v = 0, \quad (\Omega - \alpha)^{n-1}(t^{n-1}v) = (\Omega - \alpha)^{n-1}u.
\]

For then we write \( u = t^{n-1}v + (u - t^{n-1}v) \) and use the inductive assumption. By the remark made at the beginning of the proof it is obvious that the function

\[
v(x, t) = \frac{(-1)^{n-1}}{(n-1)!} (\Omega - \alpha)^{n-1}u(x, t)
\]

satisfies (3).

Having proved the theorem for \( p(\Omega) = (\Omega - \alpha)^n \), we can prove the general case of \( p(\Omega) = \prod_{i=1}^{k} (\Omega - \alpha_i)^n \) by induction on \( k \), in exactly the same way as in the case of \( p(\Delta) \) (see [2]).

Remark. That the representation (2) is unique, is easily seen by applying \( \prod_{i=1}^{k} (\Omega - \alpha_i)^n (\Omega - \alpha_j)^m m + n_j - 1, \ldots, 1, 0 \) to both sides of (2).

Corollary. The solutions \( u(x, t) \) of \( p(\Omega)u = 0 \) have the following properties:

(a) \( u(x, t) \) is infinitely differentiable in \( (x, t) \),

(b) for every \( t_0 \), \( u(x, t_0) \) is analytic in \( x \), and

(c) for every \( x_0 \), \( u(x_0, t) \) belongs to the second class of Holmgren (see [3]).

The corollary follows from Theorem 1 and the fact that metacaloric functions possess the properties (a), (b), (c).

2. Lemma 1. Let \( u(x, t) \) be a solution of \( (\Omega - \alpha)u(x, t) = 0 \) in the strip \( t_1 \leq t \leq t_2 \) and

\[
|u(x, t)| \leq Me^{Kx^2} \quad (t_1 \leq t \leq t_2).
\]

Then for every \( \epsilon > 0 \), there exists a constant \( M' = M'(\epsilon, N, M) \) such that for \( t, t_1 + \epsilon \leq t \leq t_2 \),

\[
\left| \frac{\partial^k u(x, t)}{\partial x^{k-i} \partial t^i} \right| \leq M'e^{2Kx^2} \quad (0 \leq j \leq k, 0 \leq k \leq N).
\]

Proof. It is sufficient to prove that for every \( t_0, t_1 \leq t_0 < t_2 \) and for every \( t \) which satisfies \( t_0 + \epsilon \leq t \leq t_0 + 1/8K - \epsilon, t \leq t_2 \),

\[
\left| \frac{\partial^k u(x, t)}{\partial x^{k-i} \partial t^i} \right| \leq M'e^{2Kx^2} \quad (0 \leq j \leq k, 0 \leq k \leq N)
\]

where \( M' \) depends on \( \epsilon, N, M \).
Without loss of generality, we may assume $t_0 = 0$. Consider the function

$$v(x, t) = \frac{e^{-at}}{2(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t} u(\xi, 0) d\xi$$

and apply to it the operator $\partial^k/\partial x^k - i\partial t$. Substituting $x-\xi = -2t^{1/2}s$ we have

$$\left| \frac{\partial^k v(x, t)}{\partial x^k - i\partial t} \right| \leq \int_{-\infty}^{\infty} \left| \frac{\partial^k}{\partial x^k - i\partial t} \left( \frac{e^{-at} e^{-\xi^2/4t}}{2(\pi t)^{1/2}} \right) \right| Me^{K(x+2t^{1/2}s)^2} ds$$

$$\leq M_1 \int_{-\infty}^{\infty} s^{2N} e^{-s^2} e^{2K(x^2 + 4ts^2)} ds \leq M'e^{2Ks^2}.$$ 

It remains to prove that $v(x, t) = u(x, t)$. Since the function $w = u - v$ satisfies

$$w(x, 0) = 0, \quad (\Omega - \alpha)w(x, t) = 0, \quad |w(x, t)| \leq (M + M')e^{2Ks^2},$$

it follows from [1] that $w(x, t) \equiv 0$.

3. Theorem 2. Let $u(x, t)$ be a solution of $(\Omega - \alpha)u = 0$ in the strip $t_1 \leq t \leq t_2$ and $|u(x, t)| \leq Me^{Ks^2} (t_1 \leq t \leq t_2)$. Then for every $h, t$ which satisfy $t_1 < t \leq t_2$, $t_1 < t - h$, $0 < h < 1/8K$, the following mean-value formula holds:

$$\mu(u; h) \equiv \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} u(\xi, t - h) d\xi = e^{ah} u(x, t).$$

Proof. Let $\phi(h) = 2(\pi h)^{1/2}\mu(u; h)$. Using Lemma 1 and integrating by parts, we have

$$\frac{\partial \phi}{\partial h} = \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4h} \left[ \Omega(u_\xi, t - h) + \frac{1}{2h} u(\xi, t - h) \right] d\xi = \left( \alpha + \frac{1}{2h} \right) \phi,$$

or $\phi(h) = ch^{1/2} e^{ah}$. To find $c$, observe that $\mu(u; h) = u(x, t) + o(1)$ ($h \to 0$), so that $c = 2(\pi)^{1/2} u(x, t)$ and we have proved (5).

4. Definition. A function $u(x, t)$ defined in a strip $t_1 \leq t \leq t_2$, is said to be of type $(K, n)$ (we suppose $K \neq 0$), if $\Omega^k u(x, t)$ $(k = 0, 1, \ldots, n)$ exist and satisfy

$$|\Omega^k u(x, t)| \leq Me^{Ks^2} \quad (t_1 \leq t \leq t_2, k = 0, 1, \ldots, n),$$

and of type $(0, n)$, if for every positive $\epsilon$ it is of type $(\epsilon, n)$.

Theorem 3. Let $u(x, t)$ be a solution of $p(\Omega)u = 0$, where $p(z)$
\[ \prod_{\ell=1}^{k} (z-\alpha_{\ell})^{n_{\ell}} (\sum_{i=1}^{n_{\ell}} n_{i}=n), \]\nand suppose \( u \) to be of type \((K, n-1)\) in the strip \( t_{1} \leq t \leq t_{2} \). Then, for every \( t, h \) which satisfy
\[
 t_{1} < t \leq t_{2}, \quad t_{1} < t - h, \quad 0 < h < 1/8K,
\]
\[
 \mu(u; h) = \sum_{i=1}^{k} \sum_{j=0}^{n_{i}-1} (t - h)^{i} e^{a_{i}h} u_{j}(x, t).
\]

**Proof.** Since \( u(x, t) \) is of type \((K, n-1)\), for every polynomial \( q(z) \) of degree \( \leq n - 1 \), \(|q(\Omega)u(x, t)| \leq Me^{Kx^{2}}\). Choosing \( q(z) = \prod_{\ell=1; \ell \neq j}^{k} (z-\alpha_{\ell})^{n_{\ell}}(z-\alpha_{j})^{m}, m=n_{j}-1, n_{j}-2, \ldots, 0 \) we can easily see that each \( u_{j}^{(t)}(x, t) \) in (2) is of type \((K, 0)\). Therefore, by Theorem 2
\[
 \mu(u; h) = \sum_{i=1}^{k} \sum_{j=0}^{n_{i}-1} (t - h)^{i} \mu(u_{j}; h) = \sum_{i=1}^{k} \sum_{j=0}^{n_{i}-1} (t - h)^{i} e^{a_{i}h} u_{j}(x, t).
\]

5. We now prove a generalization of Liouville's Theorem:

**Theorem 4.** Let \( p(z) \) be a polynomial of the form \( \prod_{\ell=1}^{k} (z-\alpha_{\ell})^{n_{\ell}} (\sum_{i=1}^{n_{\ell}} n_{i}=n) \) with \( \text{Re} \{ \alpha_{i} \} \geq 0 \), and let \( u(x, t) \) satisfy \( p(\Omega)u(x, t)=0 \) in the strip \(-\infty < t \leq t_{2}\). Suppose \( u(x, t) \) to be of the type \((0, n-1)\) in every strip \( t_{1} \leq t \leq t_{2} \). If \( u(x, t) \) is bounded and if
\[
 \lim_{h \to \infty} \mu(u; h) \text{ exists},
\]
then \( u(x, t) \equiv \text{const.} \)

**Proof.** Using Theorem 1 with \( D: -\infty < t \leq t_{2} \), we derive equation (6) for all positive \( h \). Write (6) in the form
\[
 \mu(u; h) = \sum_{i=1}^{n} F_{i}(x, t)\phi_{i}(h).
\]
Letting \( h \to \infty \) and comparing the behaviour at infinity of the functions \( \phi_{i}(h) \), we conclude that
\[
 \mu(u; h) = \sum_{\text{Re} \{ \alpha_{i} \}=0; \alpha_{i}, \infty}^{0} e^{a_{i}h} u_{0}(x, t) + u_{0}(x, t).
\]
The sum on the right-hand side in (9) is an almost periodic function in the sense of Bohr, and since its limit (as \( h \to \infty \)) exists, it must vanish. We obtain
\[
 \mu(u; h) = u_{0}(x, t).
\]
Noting that $\mu(u; h) = u(x, t) + o(1)$ ($h \to 0$), we conclude
\begin{equation}
\mu(u; h) = u(x, t).
\end{equation}

By [5], $\mu(u; h) = u(x, t) + h\Omega u(x, t) + o(h)$ ($h \to 0$) and together with (10), $\Omega u(x, t) = 0$. Since for caloric functions Liouville's Theorem is true (see [5]), the proof is completed.

Remark. From the proof it follows that in case that either $\text{Re} \{\alpha_i\} > 0$ or $\alpha_i = 0$ ($1 \leq i \leq k$), the assumption (7) is superfluous.

6. In this section we prove that equation (8) characterizes the $n$-metacaloric functions.

**Theorem 5.** Let $u(x, t)$ be defined in the strip $t_1 \leq t \leq t_2$ and possess second-order continuous derivatives which satisfy
\begin{equation}
\left| \begin{array}{c}
u(x, t) \\ \frac{\partial u(x, t)}{\partial t} \\ \frac{\partial u(x, t)}{\partial x} \\ \frac{\partial^2 u(x, t)}{\partial x^2}
\end{array} \right| \leq Me^{Kx^2}.
\end{equation}

Then for every $t, h$ which satisfy
\begin{equation}
t \leq t_2, \quad t_1 \leq t - h < t_2 \quad 0 < h < \frac{1}{4K},
\end{equation}
we conclude
\begin{equation}
\begin{aligned}
\Omega \mu(u; h) &= \frac{\partial \mu(u; h)}{\partial h} \\
&= \frac{1}{2(\pi h)^{1/2}} \int_{\xi = -\infty}^{\xi = \infty} e^{-(x - \xi)^2/4h} \left( \Omega u + \frac{1}{2h} u \right) d\xi.
\end{aligned}
\end{equation}

Proof. As in the proof of Theorem 3,
\begin{equation}
\frac{\partial (2(\pi h)^{1/2} \mu)}{\partial h} = \int_{\xi = -\infty}^{\xi = \infty} e^{-(x - \xi)^2/4h} \left( \Omega u + \frac{1}{2h} u \right) d\xi.
\end{equation}

Since on the other hand
\begin{equation}
\frac{\partial (2(\pi h)^{1/2} \mu)}{\partial h} = 2(\pi h)^{1/2} \frac{\partial \mu}{\partial h} + \frac{2\pi^{1/2}}{2h^{1/2}} \mu,
\end{equation}
we conclude
\begin{equation}
\frac{\partial \mu}{\partial h} = \frac{1}{2(\pi h)^{1/2}} \int_{\xi = -\infty}^{\xi = \infty} e^{-(x - \xi)^2/4h} \Omega u(\xi, t - h) d\xi = \Omega \mu.
\end{equation}

Suppose that each $\Omega^k u(x, t)$ ($k = 0, 1, \cdots, n - 1$) satisfies the assumptions of Theorem 5. We then have
\begin{equation}
\Omega^k \mu(u; h) = \frac{1}{2(\pi h)^{1/2}} \int_{\xi = -\infty}^{\xi = \infty} e^{-(x - \xi)^2/4h} \Omega^k u(\xi, t - h) d\xi
\end{equation}
\begin{equation}
= \Omega^k u(x, t) + o(1), \quad (k = 0, 1, \cdots, n; h \to 0).
\end{equation}
Suppose $u(x, t)$ satisfies (8) with $n$ times differentiable $\phi_i(h)$. Then by Theorem 5

$$\Omega^k\mu = \sum_{i=1}^n F_i(x, t) \frac{d^k\phi_i(h)}{dh^k} \quad (k = 0, 1, \ldots, n).$$

Since this system has a nontrivial solution, we conclude that

$$\begin{vmatrix} \mu(u; h) & \phi_1(h) & \cdots & \phi_n(h) \\ \Omega\mu(u; h) & \phi_1'(h) & \cdots & \phi_n'(h) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega^n\mu(u; h) & \phi_1^{(n)}(h) & \cdots & \phi_n^{(n)}(h) \end{vmatrix} = 0. \tag{13}$$

Taking $h \to 0$ and using (12) it follows that $u$ is $n$-metacaloric. Hence

**Theorem 6.** If $u(x, t)$ is $n$-metacaloric and of type $(K, n - 1)$ in the strip $t_1 \leq t \leq t_2$, then

$$\mu(u; h) = \sum_{i=1}^n F_i(x, t)\phi_i(h) \quad (t_1 < t - h, t < t_2, 0 < h < 1/8K),$$

$\phi_i(h)$ are analytic in $h$ and $F_i(x, t)$ are indefinitely differentiable in $(x, t)$ and analytic in $x$. Conversely, if

$$\begin{vmatrix} \Omega^k u(x, t) \\ \frac{\partial}{\partial t} \Omega^k u(x, t) \\ \frac{\partial}{\partial x} \Omega^k u(x, t) \end{vmatrix}, \quad \frac{\partial^2}{\partial x^2} \Omega^k u(x, t) \leq Me^{Kx^2} \quad (k = 0, \ldots, n - 1) \tag{14}$$

and if (8) holds with $n$ times differentiable $\phi_i(h)$ and for $h$ sufficiently small, then $u(x, t)$ is $n$-metacaloric and of type $(K, n - 1)$.

**7. Lemma 2.** Let $u(x, t)$ be $n$-metacaloric and of type $(K, n - 1)$ in the strip $t_1 \leq t \leq t_2$. Then (14) holds in any interval $t_1 + \epsilon \leq t \leq t_2$ and with $2K$ in place of $K$.

**Proof.** Write $u$ in the form (2). As in the proof of Theorem 3, each $u_i^{(n)}$ is of type $(K, 0)$. Apply Lemma 1 to $u_i^{(n)}$.

Suppose $u(x, t)$ to be $n$-metacaloric and of type $(K, n - 1)$. Using Lemma 2, we have

$$\Omega^k\mu(u; h) = \frac{1}{2(\pi h)^{1/2}} \int_{-\infty}^\infty e^{-(x-\xi)^2/4h} \Omega^k u(\xi, t - h) d\xi.$$ 

Since $p(\Omega)u = 0$, we also have $p(\Omega)\mu(u; h) = 0$. By Theorem 5 we get

$$p\left(\frac{\partial}{\partial h}\right) \mu(u; h) = 0. \tag{15}$$
Conversely, if $\mu(u; h)$ satisfies (15) and $u$ satisfies (14), then by Theorem 5 and (15) we get $p(\Omega)\mu(u; h) = 0$. Taking $h \to 0$, we conclude that $p(\Omega)u(x, t) = 0$. We have proved:

**Theorem 7.** If $u(x, t)$ satisfies $p(\Omega)u = 0$ and if it is of type $(K, n - 1)$, then

\[ p(\partial/\partial h)\mu(u; h) = 0 \quad (t_1 < t - h < t_2, 0 < h < 1/8K). \]

Conversely, from (14) and (16) follows $p(\Omega)u = 0$.

8. We shall derive a special form of equation (8). From (6) it is clear that the $\phi_i(h)$ are independent solutions of $p(\partial/\partial h)\phi(h) = 0$, therefore their Wronskian is different from zero. Applying $\Omega^k$ to both sides of (8) and letting $h \to 0$, we get the system

\[
\begin{align*}
\Omega^k u(x, t) = & \sum_{i=1}^{n} F_i(x, t) \frac{d^k \phi_i(0)}{dh^k} \\
& (k = 0, 1, \ldots, n - 1),
\end{align*}
\]

from which it follows that

\[
\begin{vmatrix}
\mu(u; h) & \phi_1(h) & \cdots & \phi_n(h) \\
\phi_1(0) & \phi_1(0) & \cdots & \phi_n(0) \\
\cdots & \cdots & \cdots & \cdots \\
\Omega^{n-1} u(x, t) & \phi_1^{(n-1)}(0) & \cdots & \phi_n^{(n-1)}(0)
\end{vmatrix} = 0,
\]

or

\[
\mu(u; h) = \sum_{k=0}^{n-1} c_k(h) \Omega^k u(x, t),
\]

where $c_k(h)$ are linear combinations of expressions $h^i e^{\alpha_i h}$.

**References**


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