CONVEX SETS AND NEAREST POINTS[1]

R. R. PHELPS[2]

1. **Introduction.** A well known theorem due to Motzkin [11] and extended by Busemann [3] and Jessen [7] characterizes the convexity of a closed set S in Euclidean n-space E^n in the following manner: *S is convex if and only if to each point in E^n there corresponds a unique nearest point in S.* We show here that if S is convex, the set S_z of all points having z as a nearest point in S is a *convex cone* with vertex z, while the hypothesis that S_z be merely a *cone* with vertex z (for each $z \in S$) is shown to characterize the convexity of S.

In trying to establish these results in a more general normed linear space E we find that the statement "S_z is convex whenever S is convex" is equivalent to the existence of an inner product in E when the dimension of E is at least three, while in a two-dimensional space it is equivalent to strict convexity. A theorem by Motzkin [12] leads easily to the analogous result that E is an inner product space if and only if S_z is convex for *every* $S \subset E$ (and $z \in S$).

In the concluding section we consider a nearest-point map f which assigns to each point of E a nearest point in a given closed set S. It is shown that the property "f shrinks distances whenever it exists for a closed convex set" characterizes inner product spaces of three or more dimensions. In two-dimensional spaces this property is equivalent to strict convexity and symmetry of Birkhoff's orthogonality [2]. The convexity of a closed set in E^n is shown to be characterized by the fact that its nearest-point map shrinks distances.

2. **Definitions and remarks.** Throughout this paper E will be a normed linear space and S a subset thereof. For $z \in S$, S_z will be the set $\{x: \|x-z\| = \inf_{y \in S} \|x-y\|\}$, the set of all points in E having z as a nearest point in S. It may well be that z is the only point in S_z. It is not difficult to verify that if a sequence of points in S_z converges then its limit is in S_z; hence S_z is always closed.

We will say that S is *proximinal*[3] if for each point x in E there is a

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[3] This word, a combination of *proximity* and *minimal*, was suggested by Raymond Killgrove.

point of S nearest to x, i.e., if for each point $x \in E$ there is at least one point z in S such that x is in S_z. If there is a unique such z for each x in E we will say that S is *uniquely proximinal*. Although we are not primarily interested in conditions which guarantee that S is proximinal we list some of the known ones:

(i) S is proximinal if it is compact.

(ii) Every closed set S is proximinal if E is finite dimensional.

(iii) Every closed convex set S is proximinal if E is reflexive.

3. **Cones, smoothness and strict convexity.** The following lemma has appeared before [10] but will be proved here since we use it several times.

LEMMA 3.1. *If S is convex and $z \in S$ then S_z is a cone with vertex z.*

PROOF. We can suppose $z=\phi$. It will suffice to show that if $y \in S_\phi$, then $\lambda y \in S_\phi$ for each $\lambda > 0$. Suppose $x \in S$, then $\|y\| \leqq \|y-x\|$. If $\lambda < 1$, $\|\lambda y\| + \|y-\lambda y\| = \|y\| \leqq \|y-x\| \leqq \|y-\lambda y\| + \|\lambda y - x\|$, whence $\|\lambda y\| \leqq \|\lambda y - x\|$. If $\lambda > 1$, $\lambda^{-1}x$ is in the convex set S so $\|\lambda y\| = \lambda\|y\| \leqq \lambda\|y-\lambda^{-1}x\| = \|\lambda y - x\|$. Hence $\lambda y \in S_\phi$ for each $\lambda > 0$.

A set E is *strictly convex* if the boundary of its unit cell contains no line segment, i.e., if $\|x\| = 1 = \|y\|$ and $\lambda \in]0, 1[$ imply $\|\lambda x + (1-\lambda)y\| < 1$. The following lemma characterizes strict convexity in terms useful to us.

LEMMA 3.2. *The following statements are equivalent*:

(i) *E is strictly convex.*

(ii) *For each convex set S and distinct points x and y of S, $S_x \cap S_y$ is empty.*

(iii) *Whenever a convex set S is proximinal it is uniquely proximinal.*

PROOF. (i)$\Rightarrow$(ii). If $S_x \cap S_y$ is nonempty we can suppose $\phi \in S_x \cap S_y$, so $\|x\| = \|y\|$. Now, $(1/2)(x+y) \in S$ and if E is strictly convex $\|(1/2)(x+y)\| < \|x\| = \|y\|$, a contradiction.

(ii)$\Rightarrow$(iii). This is immediate from the definitions.

(iii)$\Rightarrow$(i). Suppose E is not strictly convex, then there exist distinct points x and y such that $\|\lambda x + (1-\lambda)y\| = 1$ for each $\lambda \in [0, 1]$. The (compact) convex line segment $[x, y]$ is proximinal but not uniquely proximinal since the origin is equidistant from all points of $[x, y]$.

We say that E is *smooth* if its unit cell has a unique supporting hyperplane at each of its boundary points. (A hyperplane is a maximal proper closed linear variety.) We give a partial converse to Lemma 3.1 in the following lemma (stated but not proved in [**10**]).

LEMMA 3.3. *Suppose S is closed and proximinal and E is smooth. Then S is convex if for each $z \in S$, S_z is a cone with vertex z.*

PROOF. Suppose S is not convex. Then there exist points u and v in S such that $]u, v[\subset E\sim S$. Let $x=(1/2)(u+v)$. Since S is proximinal $x\in S_z$ for some $z\in S$ and we can suppose without loss of generality that $z=\phi$. Let H be the unique hyperplane supporting $N_{\|x\|}x\equiv\{y: \|y-x\|<\|x\|\}$ at ϕ and let H' be the open halfspace determined by H which contains $N_{\|x\|}x$. Not both u and v are in the convex set $E\sim H'$ since x is not. Suppose $u\in H'$. We will show that for some $\lambda>0$, $u\in N_{\|\lambda x\|}\lambda x=\lambda N_{\|x\|}x$ so that $\|\lambda x\|>\|\lambda x-u\|$ and hence $\lambda x\notin S_\phi$, contradicting the assumption that S_ϕ is a cone with vertex ϕ.

Suppose that for every $\lambda>0$, $u\notin\lambda N_{\|x\|}x$. Then $\alpha u\notin_{\|x\|}x$ for each $\alpha\in]0, 1[$ and hence the convex set $[u, \phi[$ is disjoint from $N_{\|x\|}x$. There exists a hyperplane G separating $[u, \phi[$ from $N_{\|x\|}x$ which necessarily supports $N_{\|x\|}x$ at ϕ. But u is in the closed halfspace determined by G which does not contain $N_{\|x\|}x$ and hence $G\neq H$, contradicting the fact that E is smooth. Thus for some $\lambda>0$, $u\in\lambda N_{\|x\|}x$, which was to be shown.

The assumption of smoothness of E in Lemma 3.3 is a necessary one, since it is not difficult to show that the statement of the lemma, with smoothness omitted, *implies* that E is smooth.

Since every closed subset of (smooth) E^n is proximinal, Lemmas 3.1 and 3.3 combine to prove the following characterization of convexity.

THEOREM 3.4. *A closed set S in E^n is convex if and only if for each $z\in S$, S_z is a cone with vertex z.*

4. **Convex cones and inner products.** We say that E is an *inner product space* if E admits an inner product (x, y) such that $\|x\| =(x, x)^{1/2}$. If E is an inner product space it is easily verified that for points x, y and z in E and $\lambda\in R$ the following identity holds:

$$
\begin{aligned}
&\|z-[\lambda x+(1-\lambda)y]\|^2 \\
&\qquad =\lambda\|x-z\|^2-\lambda(1-\lambda)\|x-y\|^2+(1-\lambda)\|z-y\|^2.
\end{aligned}
\tag{1}
$$

(In fact [8] E admits an inner product if and only if (1) holds true.)

In particular, we can see that an inner product space is strictly convex by setting $z=\phi$, $\|x\|=1=\|y\|$ and $\lambda\in]0, 1[$ in (1).

LEMMA 4.1. *If E is an inner product space and $z\in S$ then S_z is convex.*

PROOF. Suppose $z=\phi$ and that $x\in S_\phi$ and $y\in S_\phi$. Let $w=\lambda x+(1-\lambda)y$, $\lambda\in]0, 1[$. Then if v is any point of S, $\|x\|\leqq\|x-v\|$ and $\|y\|\leqq\|y-v\|$ while, by (1), we have

$$
\begin{aligned}
\|w\|^2&=\lambda\|x\|^2-\lambda(1-\lambda)\|x-y\|^2+(1-\lambda)\|y\|^2\\
&\leqq\lambda\|x-v\|^2-\lambda(1-\lambda)\|x-y\|^2+(1-\lambda)\|v-y\|^2=\|w-v\|^2,
\end{aligned}
$$

so $\|w\| \leqq \|w-v\|$ and hence $w \in S_\phi$.

Motzkin [12] has proved the following interesting result: *Suppose that E is two-dimensional. Then E is an inner product space if and only if for each set S and $z \in S$, S_z is convex.* Since a normed linear space is an inner product space if and only if each two-dimensional subspace has an inner product [8], Motzkin's result leads easily to the sufficiency portion of the following theorem.

THEOREM 4.2. *A normed linear space E is an inner product space if and only if for each set $S \subset E$ and $z \in S$, S_z is convex.*

Demanding convexity of S_z only when S itself is convex leads to the following result, closely related to Theorem 4.2.

THEOREM 4.3. *Suppose that the dimension of E is at least three [resp. equal to two]. Then E is an inner product space [resp. strictly convex] if and only if for each convex set S and $z \in S$, S_z is convex.*

The portions of Theorem 4.3 which are as yet unproved follow from the next three lemmas. The idea used in the proof of the following lemma is due to James [6, Theorem 2].

LEMMA 4.4. *Suppose the dimension of E is at least three. Then E is an inner product space provided S_z is convex for each convex set $S \subset E$ and $z \in S$.*

PROOF. If x_1 and x_2 are any two linearly independent points of E there exist hyperplanes H^1 and H^2 such that $x_1 \in H^1_\phi$ and $x_2 \in H^2_\phi$ ($H^i = h_i^{-1}(0)$, where h_i is a continuous linear functional such that $h_i(x_i) = \|x_i\|$ and $\|h_i\| = 1$ [1, p. 55]). By Lemma 3.1, H^1_ϕ and H^2_ϕ are convex *cones* with vertex ϕ and hence $\alpha_i x_i \in H^i_\phi$ if $\alpha_i \geqq 0$ ($i = 1, 2$). But $y \in H^1$ if and only if $-y \in H^1$, so $\|(-x_1) - y\| = \|x_1 - (-y)\| \geqq \|x_1\| = \|-x_1\|$, which shows that $-x_1 \in H^1_\phi$ and therefore $\alpha_i x_1 \in H^1_\phi$ for $\alpha_1 \in R$. Similarly, $\alpha_2 x_2 \in H^2_\phi$ for $\alpha_2 \in R$. Thus, if we let $G = H^1 \cap H^2$, G_ϕ is also a convex cone with vertex ϕ and hence contains $\alpha_1 x_1 + \alpha_2 x_2$ for $\alpha_1 \in R$ and $\alpha_2 \in R$. Since H^1 and H^2 are hyperplanes, E is the direct sum of G and the two-dimensional sub-space F spanned by x_1 and x_2. Therefore, if $z \in E$, $z = (\alpha_1 x_1 + \alpha_2 x_2) - g$, where $-g \in G$, and $\alpha_1 \in R$, $\alpha_2 \in R$. Letting $f(z) = \alpha_1 x_1 + \alpha_2 x_2$ we see that f is a projection onto F and since $f(z) \in G_\phi$ and $g \in G$, $\|f(z)\| \leqq \|f(z) - g\| = \|z\|$ and therefore $\|f\| = 1$.

Thus, we can always find a projection of norm one on any two-dimensional subspace of E and therefore it is possible to define an inner product in any three-dimensional subspace of E [9, Theorem 3]. This, however, implies that we can define an inner product in E itself [8].

By a less direct argument than the above it can be shown that the conclusion still holds if the hypothesis "S_z is convex whenever $z\in S$ and S is convex" be replaced by "L_z is convex whenever L is a line and $z\in L$."

LEMMA 4.5. *If L_z is convex for each line L and $z\in L$ then E is strictly convex.*

PROOF. If E is not strictly convex there exist distinct points x and y such that $\|\lambda x+(1-\lambda)y\|=1$ for each $\lambda\in[0,1]$. Let $L=\{\lambda x+(1-\lambda)y:\lambda\in R\}$. If $\lambda>1$, $\|\lambda x+(1-\lambda)y\|\geqq\lambda\|x\|-|(1-\lambda)|\,\|y\|=1$ while if $\lambda<0$, $\|\lambda x+(1-\lambda)y\|\geqq(1-\lambda)\|y\|-|\lambda|\,\|x\|=1$. Thus, $(1/2)x\in L_{(1/2)(x+y)}$ since if z is any point of L, $\|z-(1/2)x\|\geqq\|z\|-(1/2)\|x\|\geqq(1/2)=(1/2)\|y\|=\|(1/2)(x+y)-(1/2)x\|$. Similarly, $(1/2)y\in L_{(1/2)(x+y)}$. Further, $x+(1/2)y\in L_{(1/2)(x+y)}$, for if $z\in L$, then $x+y-z\in L$ and hence

$$\begin{aligned}\|(x+(1/2)y)-(1/2)(x+y)\|&\\ &=\|(1/2)x\|=\|(1/2)(x+y)-(1/2)y\|\\ &\leqq\|(x+y-z)-(1/2)y\|=\|(x+(1/2)y)-z\|.\end{aligned}$$

Since $L_{(1/2)(x+y)}$ is assumed to be convex,

$$(1/2)[x+(1/2)y]+(1/2)[(1/2)x]=(3/4)x+(1/4)y\in L_{(1/2)(x+y)},$$

which is impossible, $(3/4)x+(1/4)y$ itself being a point of L.

LEMMA 4.6. *Suppose E is strictly convex and of dimension two. Then if S is convex and $z\in S$, S_z is a convex cone with vertex z.*

PROOF. By Lemma 3.1, S_z is a cone with vertex z, so it remains only to show that S_z is convex. Suppose $z=\phi$ and suppose $x\in S_\phi$ and $y\in S_\phi$; we must show that $[x,y]\subset S_\phi$.

Let K be the closed convex cone generated by all the rays passing from ϕ through points of $[x,y]$. Then $K\cap S=\{\phi\}$, for if $w\in K\cap S$ there exists $\lambda\in]0,1]$ such that λw is in the closed triangle ϕxy and since S is convex, $\lambda w\in S$. Let $u=(\|x\|+\|y\|)^{-1}(\|y\|x+\|x\|y)$. Then $u\in]x,y[$ and hence λw is in the closed triangle ϕux, say. (Otherwise λw is in ϕuy.) But if αu, $\alpha\in[0,1]$, is any point of side $[\phi,u]$, $\|x-\alpha u\|\leqq\|x\|$. Consequently, $\|x-\lambda w\|\leqq\|x\|$ and so $x\in S_{\lambda w}$. By strict convexity and Lemma 3.2, $\lambda w=\phi$ and therefore $w=\phi$.

Now suppose $z\in[x,y]$ and $v\in S$. Then $[z,v]$ must intersect $\{\lambda x:\lambda\geqq 0\}$ or $\{\lambda y:\lambda\geqq 0\}$; say $\lambda x\in[z,v]$, $\lambda\geqq 0$. Then $\lambda x\in S_\phi$ (since S_ϕ is a cone) and $\|z\|\leqq\|z-\lambda x\|+\|\lambda x\|\leqq\|z-\lambda x\|+\|\lambda x-v\|=\|z-v\|$. Since this holds for arbitrary $v\in S$, $z\in S_\phi$.

5. **The nearest-point map.** If a closed set S in E is proximinal we can define a function f from E onto S as follows: If $x \in E$ let $f(x)$ be a point of S such that $x \in S_{f(x)}$. It is clear that f, called a *nearest-point map for* S, exists if and only if S is proximinal, and that f is unique if and only if S is uniquely proximinal. We say that f *shrinks distances* if $\|f(x)-f(y)\| \leqq \|x-y\|$ whenever $x, y \in E$. We will say that E *has the property* P if a nearest-point map shrinks distances whenever it exists for a closed convex set $S \subset E$. The following theorem is well known, but a proof is included for completeness.

THEOREM 5.1. *Each inner product space E has the property* P.

PROOF. Suppose a nearest-point map f exists for a closed convex set S. Since E is strictly convex Lemma 3.2 implies that f is unique. Suppose $x \in E$ and $y \in E$ and that $f(x)=\phi$. Let H be the hyperplane through ϕ which is orthogonal to $f(y)$ and let J be the open half-space determined by H which contains $f(y)$. Let K be the open half-space determined by $H+f(y)$ which contains ϕ. If $x \in J$ there exists $\alpha>0$ such that $\|x\|>\|x-\alpha f(y)\|$. Pick $\lambda>0$ such that $\lambda\alpha=1/2$, then $\|\lambda x\|>\|\lambda x-(1/2)f(y)\|$. But, since $(1/2)f(y) \in S$, this contradicts the fact that $f(x)$, and hence $f(\lambda x)$, is the origin. We conclude that $x \notin J$ and an entirely similar argument shows that $y \notin K$. Thus, $\|x-y\|$ is no less than the width of $J \cap K$, and this is equal to $\|f(y)\|$.

Birkhoff [2] has defined a type of orthogonality which is meaningful in a general normed linear space E and which coincides with the usual notion in an inner product space. If $x \neq 0$ we say that *y is orthogonal to x* (written $y \perp x$) if $\|y-\lambda x\| \geqq \|y\|$ for each $\lambda \in R$. Note that this is equivalent to saying that $y \in (Rx)_\phi$, where $Rx=\{\lambda x : \lambda \in R\}$ is the line determined by x and ϕ. We say that *orthogonality is symmetric* if $x \perp y$ implies $y \perp x$. Day [4, Theorem 6.4] and James [6, Theorem 1] have independently proved that *a normed linear space of dimension at least three is an inner product space if and only if orthogonality is symmetric.* We use this fact in proving the following theorem.

THEOREM 5.2. *Suppose that the dimension of E is at least three [resp. equal to two]. Then E is an inner product space [resp. strictly convex and orthogonality is symmetric] if and only if E has the property* P.

The proof is contained in Theorem 5.1 and the following succession of remarks and lemmas.

LEMMA 5.3. *If E has the property* P *then E is strictly convex.*

Since the proof of this lemma is quite straightforward, it will be omitted.

LEMMA 5.4. *If E has the property* P *then orthogonality in E is symmetric.*

PROOF. By Lemma 5.3, E must be strictly convex and hence a nearest-point map is unique whenever it exists for a closed convex set. Suppose that neither y nor x is the origin and that $y \perp x$. The line Rx is uniquely proximinal and the nearest-point map f exists for Rx. Since $Ry \subset (Rx)_\phi$, $f(\lambda y) = \phi$ for any $\lambda \in R$. Now E has the property P, so $\|x\| = \|f(x) - f(\lambda y)\| \leqq \|x - \lambda y\|$ for any $\lambda \in R$, i.e., $x \in (Ry)_\phi$ or $x \perp y$. Thus, orthogonality is symmetric.

If the dimension of E is at least three, the Day-James theorem mentioned above, together with Lemma 5.4, proves that if E has property P it is an inner-product space.

LEMMA 5.5. *Suppose that E is two-dimensional. If E is strictly convex and orthogonality is symmetric then E has the property* P.

PROOF. Suppose the nearest-point map f exists for a closed convex set S and suppose $x, y \in E$. We can assume that $f(x) = \phi$. There exists a point $z \neq \phi$ such that $z \perp f(y)$ and, since E is strictly convex, $w \perp f(y)$ implies $w \in Rz$ [5, Theorem 4.3]. Let J be the open half-space determined by Rz which contains $f(y)$ and let K be the open half-space determined by $Rz + f(y)$ which contains ϕ. If $x \in J$ there exists a unique $\alpha \in R$ such that $x - \alpha f(y) \perp f(y)$ [5]. Now, $\alpha > 0$ since $x - \alpha f(y) \in Rz$ and x is on the same side of Rz as is $f(y)$. Thus, using strict convexity again, $\|x - \alpha f(y)\| < \|x\|$. As in the proof of Theorem 5.1 we conclude that $x \notin J$. A similar argument shows that $y \notin K$. Thus, $\|x - y\|$ is no less than the width of $K \cap J$. Now, by symmetry of orthogonality, $f(y) \perp z$ and so the distance from $f(y)$ to Rz is attained at ϕ. Hence the distance from $Rz + f(y)$ to Rz (which is the width of $J \cap K$) is equal to $\|f(y)\|$ and therefore $\|f(y)\| \leqq \|x - y\|$.

It is not hard to see that neither strict convexity nor symmetry can be omitted in Lemma 5.5, since P implies both and there exist examples showing that neither implies the other.

The following theorem shows that the "shrinking" property of nearest-point maps is pretty well restricted to those which exist for convex sets.

THEOREM 5.6. *Suppose that E is strictly convex and that a nearest-point map f exists for the closed set $S \subset E$. Then S is convex iff f shrinks distances.*

PROOF. If S is not convex there exist distinct points x and y of S such that $]x, y[\subset E \sim S$. Letting $z = (1/2)(x+y)$ we see that one of

$\|x-f(z)\|$, $\|y-f(z)\|$ is greater than $(1/2)\|x-y\|$. (This is obvious if $f(z)$ is x or y, while if $f(z)\neq x$, y and neither $\|x-f(z)\|$ nor $\|y-f(z)\|$ is greater than $(1/2)\|x-y\|$, strict convexity implies that $\|x-y\| < \|x-f(z)\| + \|y-f(z)\| \leqq \|x-y\|$, a contradiction.) Suppose, then, that $\|x-f(z)\| > (1/2)\|x-y\| = \|x-z\|$. Since $f(x)=x$, this contradicts the assumption that f shrinks distances. We get the same contradiction if $\|y-f(z)\| > (1/2)\|x-y\|$, hence S must be convex.

A simple two-dimensional example can be constructed to show that we need to assume strict convexity in the above theorem.

Since every closed subset of Euclidean n-space E^n is proximinal, Theorems 5.1 and 5.6 combine to give the following corollary.

COROLLARY 5.7. *Let f be a nearest-point map for the closed set $S\subset E^n$. Then S is convex if and only if f shrinks distances.*

REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.
2. G. Birkhoff, *Orthogonality in linear metric spaces*, Duke Math. J. vol. 1 (1935) pp. 169–172.
3. H. Busemann, *Note on a theorem of convex sets*, Matematisk Tidsskrift B (1947) pp. 32–34.
4. M. M. Day, *Some characterizations of inner-product spaces*, Trans. Amer. Math. Soc. vol. 62 (1947) pp. 320–337.
5. R. C. James, *Orthogonality and linear functionals*, Trans. Amer. Math. Soc. vol. 61 (1947) pp. 265–292.
6. ———, *Inner products in normed linear spaces*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 559–566.
7. B. Jessen, *Two theorems on convex point sets*, Matematisk Tidsskrift. B (1940) pp. 66–70 (Danish).
8. P. Jordan and J. von Neumann, *On inner products in linear spaces*, Ann. of Math. (2) vol. 36 (1935) pp. 719–723.
9. S. Kakutani, *Some characterizations of Euclidean space*, Jap. J. Math. vol. 16 (1939) pp. 93–97.
10. V. L. Klee, *A characterization of convex sets*, Amer. Math. Monthly vol. 56 (1949) pp. 247–249.
11. Th. Motzkin, *Sur quelques propriétés caractéristiques des ensembles convexes*, Atti Acad. Naz. Lincei Rend 6 vol. 21 (1935) pp. 562–567.
12. ———, *Sur quelques propriétés caractéristiques des ensembles bornés non convexes*, Atti Acad. Naz. Lincei Rend 6 vol. 21 (1935) pp. 773–779.

UNIVERSITY OF CALIFORNIA, LOS ANGELES AND
UNIVERSITY OF WASHINGTON