FIXED POINTS OF SYMMETRIC PRODUCT MAPPINGS

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1. Introduction. Let $X$ be a topological space and $X^n$ the $n$th cartesian product in the usual topology. Let $G$ be any group of permutations of the letters $[1, \cdots, n]$. Then $G$ can be considered as a group of homeomorphisms on $X^n$ by defining, for $g \in G$ and $(x_1, \cdots, x_n) \in X^n$, $g(x_1, \cdots, x_n) = (x_{g(1)}, \cdots, x_{g(n)})$. The orbit space under the group (in the identification topology) will be denoted by $X^n/G$ and called a $G$-product of $X$. Let $\eta: X \rightarrow X^n/G$ be the identification map.¹

We will consider continuous functions $f: X \rightarrow X^n/G$ and will say that an element $x \in X$ is fixed under $f$ if $f(x)$ has $x$ as one of its coordinates. The purpose of this paper is to associate to each $G$-product map a Lefschetz number (an integer depending upon the homotopy class of the function) with the property that if this number is not zero, then the function has a fixed point [2]. The Lefschetz number will be defined only in case $X$ is a polyhedron. An application will be given regarding spaces with a finite group of operators.

2. $G$-products and a special homology homomorphism. Let the $i$th projection of $X^n$ onto $X$ be denoted by $\pi_i$. It is clear that the group and the projections are interrelated by $\pi_i g(z) = \pi_{g(i)} z$ for $z \in X^n$ and $g \in G$.

In case $X$ is metric, then using the usual euclidean metric on $X^n$, $G$ becomes a group of isometries on $X^n$. A metric may be introduced in $X^n/G$ by defining

$$d(\eta(z), \eta(z')) = \inf \{ d(z, gz') \mid g \in G \}$$

where $z, z' \in X^n$. It is convenient also to introduce a metric-like function on $X \times (X^n/G)$ defined by

$$\omega(x, \eta(z)) = \inf \{ d(x, \pi_i(z)) \mid i = 1, \cdots, n \}$$

where $x \in X$ and $z \in X^n$. Then $\omega$ is continuous and satisfies the inequality:

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\[ \omega(x, y) \leq \omega(x, y') + d(y', y) \]

for any \( x \in X \) and \( y, y' \in X^n/G \).

Now let \( X = |K| \) be a polyhedron. Denote the set of vertices of \( K \) by \( V \) and the set simplexes of \( K \) by \( S \). Assume that \( K \) is an ordered complex, that is to say that a partial ordering \( \leq \) is defined on \( V \) which is a linear ordering on any subset of \( V \) in \( S \). A triangulation \( K^n \) of \( X^n \) may be obtained as follows [1, p. 67]: The set of vertices of \( K^n \) is \( V^n \). Let \( \pi_i: V^n \to V \) be the \( i \)th projection. For \( w, w' \in V^n \), define \( w \leq w' \) if \( \pi_i w \leq \pi_i w' \) for all \( i = 1, \ldots, n \). A subset \( t = (w_0, \ldots, w_p) \) of \( V^n \) is a simplex of \( K^n \) if it is linearly ordered and the vertices \( \pi_i w_0, \ldots, \pi_i w_p \) span a simplex of \( K \) for \( i = 1, \ldots, n \) (the vertices \( \pi_i w_0, \ldots, \pi_i w_p \) need not be distinct). It follows from the definition of \( K^n \) that the projections are simplicial and that \( G \) is a group of order preserving functions on \( K^n \). Since each \( g \in G \) is order preserving on \( V^n \), we have that if \( (w, gw) \) is a simplex of \( K^n \), then \( w \leq gw \leq g^2w \leq \cdots \leq g^k w = w \) for some integer \( k \), and hence \( gw = w \).

Let \( Sd(K^n) \) denote the first barycentric subdivision of \( K^n \) (the set \( B \) of vertices of \( Sd(K^n) \) consists in all barycenters \( b_t \) of simplexes \( t \) of \( K^n \)). The group \( G \) operates on \( Sd(K^n) \) by \( gb_t = b_{\pi t} \) and the simplicial map \( \phi: Sd(K^n) \to K^n \) (which associates to \( b_t \) the least vertex of \( t \)) commutes with each \( g \in G \). Furthermore, if \( (b_t, b_{t'}) \) and \( (b_i, gb_{t'}) \) are both simplexes of \( Sd(K^n) \), then \( gb_t = b_i \). For if \( (b_t, b_{t'}) \) and \( (g^{-1}b_t, b_{t'}) \) are both simplexes of \( Sd(K^n) \) then \( t \) and \( g^{-1}(t) \) are both faces of \( t' \). If we let \( w \) denote any vertex of \( t \), then \( (w, g^{-1}w) \) is a simplex of \( K^n \). By a previous argument, \( w = gw \). Hence \( t = gt \) and \( gb_t = b_i \).

A triangulation \( K(n, G) \) for \( X^n/G \) can now be defined. The set \( A \) of vertices of \( K(n, G) \) is the set of equivalence class of elements of \( B \) under \( G \). A subset \( (a_0, \ldots, a_p) \) of \( A \) form a simplex of \( K(n, G) \) if there exists \( b_i \in a_i \) so that \( (b_0, \ldots, b_p) \) is a simplex of \( Sd(K^n) \). If another choice is made, say \( b'_i \in a_i \) so that \( (b'_0, \ldots, b'_p) \) is a simplex of \( Sd(K^n) \), then \( b'_i = g_i b_i \) for some \( g_i \in G \), \( i = 0, \ldots, p \). For any \( i \) we have \( (b_i, b_p) \) and \( (g_i b_i, g_p b_p) \) simplexes of \( Sd(K^n) \), and therefore \( (b_i, b_p) \) and \( (b_i, g_i^{-1} g_p b_p) \) are simplexes of \( Sd(K^n) \). By a previous argument, we have \( g_i^{-1} g_p b_i = b_i \), and hence \( g_p b_i = g_i b_i \). Therefore

\[
(b'_0, \ldots, b'_p) = (g_0 b_0, \ldots, g_p b_p) = (g_p b_0, \ldots, g_p b_p) = g_p (b_0, \ldots, b_p).
\]

The \( p \)-simplexes of \( K(n, G) \) are therefore in one-to-one correspondence with the equivalence classes of \( p \)-simplexes of \( Sd(K^n) \).

Consider now the integral chain groups defined on oriented simplexes of these complexes:
We wish to complete the rectangle with a homomorphism \( \mu: C_p(K(n, G)) \to C_p(K) \) so that the diagram commutes (\( \sum_{i=1}^{n} \pi_i \phi \) is the sum of the projection chain maps). Let \( t = (a_0, \cdots, a_p) \) be a generator of \( C_p(K(n, G)) \). Choose any \( \sigma = (b_0, \cdots, b_p) \) generator of \( C_p(Sd(K^n)) \) for which \( \eta \phi(\sigma) = t \). Then define

\[
\mu(t) = \sum_{i=1}^{n} \pi_i \phi_i(\sigma).
\]

The definition is independent of the choice of \( \sigma \). For if \( \eta \phi(\sigma') = \eta \phi(\sigma) = t \), then \( \sigma' = g \sigma \) for some \( g \in G \), and

\[
\sum_{i=1}^{n} \pi_i \phi_i(g \sigma) = \sum_{i=1}^{n} \pi_i g \phi_i(\sigma) = \sum_{i=1}^{n} \pi_{g(\sigma)} \phi_i(\sigma) = \sum_{i=1}^{n} \pi_i \phi_i(\sigma).
\]

Also, \( \mu \) is a chain map, since \( \eta \phi, \phi_i \), and \( \sum_{i=1}^{n} \pi_i \phi_i \) each commute with the boundary operator, and hence \( \mu \) induces \( \mu^*_\phi: H_p(Xn/G) \to H_p(X) \).

From the commutativity of the above mapping diagram, we have \( \sum_{i=1}^{n} \pi_i \phi_i \phi_j(f\sigma) = \sum_{i=1}^{n} \pi_i (f \circ \eta \phi_i) \phi_j(\sigma) = \sum_{i=1}^{n} \pi_{(f \circ \eta \phi_i)} \phi_j(\sigma) = \sum_{i=1}^{n} \pi_i \phi_i \phi_j(\sigma) \).

3. **The topological invariance of \( \mu \).** Let \( X \) and \( Y \) be any two spaces and \( F: X \to Y \) a continuous function. Then \( F \) induces a map \( \bar{F}: X^n/G \to Y^n/G \) defined as follows: Let \( F^n: X^n \to Y^n \) be given by \( F^n(x_1, \cdots, x_n) = (Fx_1, \cdots, Fx_n) \). Then \( F^n \) is equivariant with respect to \( G \) and hence induces a continuous map \( \bar{F} \) on the quotient spaces. If \( F_1: X \to Y \) and \( F_2: Y \to Z \), then clearly the function induced on the corresponding \( G \) products by the composite \( F_2F_1 \) is the composite \( \bar{F}_2\bar{F}_1 \). Also, if \( F_0 \) and \( F_1 \) are two maps of \( X \) into \( Y \) which are homotopic, then \( \bar{F}_0 \) and \( \bar{F}_1 \) are homotopic. If \( F: X \to Y \) is a homeomorphism, then \( \bar{F} \) is also.
Let $X = |K|$ and $Y = |L|$ be polyhedra where $K$ and $L$ are ordered complexes. Then the complexes $K(n, G)$ and $L(n, G)$ are defined as before and depend upon the vertex ordering of $K$ and $L$, respectively. Let $f: K \to L$ be a simplicial order preserving function. Then $f^n$ can be defined on the vertices of $K^n$ into the vertices of $L^n$ by $f^n(v_1 \times \cdot \cdot \cdot \times v_n) = (f_1 v_1 \times \cdot \cdot \cdot \times f_n v_n)$ and $f^n$ is order preserving on the vertices of $K^n$. So, if $w_0 \leq \cdot \cdot \cdot \leq w_p$ is a simply ordered set of vertices of $K^n$, $f^n w_0 \leq \cdot \cdot \cdot \leq f^n w_p$ is simply ordered. Furthermore, $\bar{\pi}_i f^n = f \pi_i$ for all $i = 1, \cdot \cdot \cdot , n$ where $\pi_i$ and $\bar{\pi}_i$ are the projections in $K$ and $L$, respectively. Hence, $f^n: K^n \to L^n$ is a simplicial map and induces $(f^n)': Sd(K^n) \to Sd(L^n)$ in the usual fashion. The map $f^n$ is equivariant with respect to $G$ and hence $(f^n)'$ is also. Therefore, $(f^n)'$ induces a simplicial function $\hat{f}: K(n, G) \to L(n, G)$, and the continuous function induces on $|K|^n/G$ into $|L|^n/G$ by $f$ is the same as the simplicial map $\hat{f}$. If we denote by $\phi: Sd(K^n) \to K^n$ and $\bar{\phi}: Sd(L^n) \to L^n$ the order preserving subdivision maps, then we have $f^n \phi = \bar{\phi}(f^n)'$. If $\eta: Sd(K^n) \to K(n, G)$ and $\bar{\eta}: Sd(L^n) \to L(n, G)$ are the identification maps, then $\bar{\eta}(f^n)' = \bar{\eta}(f^n)$.

**Lemma 1.** If $f: K \to L$ is an order preserving simplicial map, then $f_* \mu_K = \mu_L f_*$. 

**Proof.** Take $\sigma$ a generator of $C_p(K(n, G))$ and let $t$ be a generator of $C_p(Sd(K^n))$ for which $\eta(t) = \sigma$. Then

$$f_* \mu_K(\sigma) = \hat{f}_* \sum_{i=1}^n \pi_i \hat{\phi}_i(t)$$

$$= \sum_{i=1}^n \bar{\pi}_i \hat{f}_* \phi_i(t)$$

$$= \sum_{i=1}^n \bar{\pi}_i \hat{\phi}_i (f^n)'(t)$$

$$= \mu_L(\hat{f}_* \sigma)$$

since $\bar{\eta}((f^n)'(t)) = \bar{\eta}(t) = \hat{f}(\sigma)$. Q.E.D.

**Lemma 2.** If $f: K \to L$ is any simplicial function (not necessarily order preserving), then $f_* \mu_K = \mu_L f_*$. 

**Proof.** Let $K'$ and $L'$ be the barycentric subdivision of $K$ and $L$, respectively. Let $\alpha: K' \to K$ and $\beta: L' \to L$ be the simplicial maps which associate to each barycenter of a simplex the least vertex of the simplex. Then $\alpha$ and $\beta$ are order preserving. Let $f': K' \to L'$ be the simplicial map induced by $f$. Then $f'$ is order preserving, and we have
\[
\begin{array}{cccc}
H_p(K) & \xleftarrow{\alpha_*} & H_p(K') & \xrightarrow{f'_*} H_p(L') & \xrightarrow{\beta_*} H_p(L) \\
\uparrow \mu_{K*} & & \uparrow \mu_{K'*} & & \uparrow \mu_{L'}* \\
H_p(K(n, G)) & \xleftarrow{\tilde{\alpha}_*} & H_p(K'(n, G)) & \xrightarrow{\tilde{f'}*} H_p(L'(n, G)) & \xrightarrow{\tilde{\beta}_*} H_p(L(n, G)) \\
\end{array}
\]

By Lemma 1, each rectangle commutes. It is clear that \( \alpha_* \) is an isomorphism and that \( f_*=\beta_*/\alpha_*^{-1} \). Since \( \alpha \) and \( \beta \) are homotopic to the identity on \(|K|\) and \(|L|\), respectively, \( \tilde{\alpha} \) and \( \tilde{\beta} \) are homotopic to the identity on \(|K(n, G)|\) and \(|L(n, G)|\). Therefore, \( \tilde{\alpha}_* \) and \( \tilde{\beta}_* \) are isomorphisms. Since \( f_\alpha \) and \( f_\beta' \) are homotopic, we have \( \tilde{f}\tilde{\alpha} \) homotopic to \( \tilde{\beta}\tilde{f}' \). Hence, \( \tilde{f}_*\tilde{\alpha}_*=\tilde{\beta}_*\tilde{f}'_* \) and \( \tilde{f}_*=\tilde{\beta}_*\tilde{f}'_*\tilde{\alpha}_*-1 \). Therefore, \( f_*\mu_{K*} = \mu_{L*}\tilde{F}_* \).

**Lemma 3.** If \( F: |K| \rightarrow |L| \) is any continuous function, then \( F_*\mu_{K*} = \mu_{L*}\tilde{F}_* \).

**Proof.** Let \( N \) be a barycentric subdivision of \( K \), sufficiently fine so that a simplicial map \( f: N \rightarrow L \) exists which is a simplicial approximation to \( F[2] \). Let \( \gamma: N \rightarrow K \) be a simplicial subdivision map. We have the diagram:

\[
\begin{array}{cccc}
H_p(K) & \xleftarrow{\gamma_*} & H_p(N) & \xrightarrow{f_*} H_p(L) \\\n\uparrow \mu_{K*} & & \uparrow \mu_{N*} & & \uparrow \mu_{L*} \\
H_p(K(n, G)) & \xleftarrow{\tilde{\gamma}_*} & H_p(N(n, G)) & \xrightarrow{\tilde{f}_*} H_p(L(n, G)) \\
\end{array}
\]

By the previous lemma, each rectangle commutes. In the top row, \( \gamma_* \) is an isomorphism and \( F_*=f_*\gamma_*^{-1} \). Since \( \gamma \) is homotopic to the identity on \(|K|\), \( \tilde{\gamma} \) is homotopic to the identity on \(|K|^{n/G}\) and \( \tilde{\gamma}_* \) is an isomorphism. Since \( F\gamma \) is homotopic to \( F \), we have \( \tilde{F}\tilde{\gamma} \) homotopic to \( \tilde{F} \). Hence \( \tilde{F}_*\tilde{\gamma}_*=\tilde{f}_* \) and \( \tilde{F}_*=\tilde{f}_*\tilde{\gamma}_*-1 \). Therefore, \( F_*\mu_{K*} = \mu_{L*}\tilde{F}_* \).

Now if we take \( F \) to be a homeomorphism, we have:

**Corollary.** The homomorphism \( \mu_* \) is independent of the triangulation.

4. **Main theorem.** As before, let \( X = |K| \) be a polyhedron and let \( f: X \rightarrow X^{n/G} \) be a continuous function. In the remainder of the paper the coefficient group for the homology groups will be the field of rational numbers. Consider the composite homomorphism

\[
H_p(X) \xrightarrow{f_p^*} H_p(X^{n/G}) \xrightarrow{\mu_p^*} H_p(X)
\]

Since \( \mu_p^*f_p^* \) is a linear transformation of a finite dimensional vector space into itself, the trace of \( \mu_p^*f_p^* \) is defined. The Lefschetz number \( \mathcal{L}(f) \) is defined to be the number \( \sum_{p=0}^{\infty} (-1)^p \) trace \( (\mu_p^*f_p^*) \). Since \( f_p^* \)
and $\mu^p_*$ are independent of the triangulation, $\mathcal{L}(f)$ is also. In case $n = 1$, then $\mu^p_*$ is the identity and $\mathcal{L}(f)$ reduces to the Lefschetz number as defined in [2].

**Theorem 1.** If $X$ is a polyhedron and $f : X \to X^n/G$ where $\mathcal{L}(f) \neq 0$, then $f$ has a fixed point.

**Proof.** Assume, by way of contradiction, that $f$ has no fixed points. Then $\omega(x, f(x)) > 0$ for all $x \in X$. Since $X$ is compact, there exists $\epsilon > 0$ such that $\omega(x, f(x)) > \epsilon$ for all $x \in X$.

Let a triangulation $\mathcal{T}$ of $X$ be chosen so small that mesh $\mathcal{T} < \epsilon/3$. Then clearly mesh $\mathcal{T} < \epsilon/3$ and mesh $\mathcal{T}(n, G) < \epsilon/3$. By the simplicial approximation theorem, there is a subdivision $\mathcal{T}_1$ of $\mathcal{T}$ and a map $h : \mathcal{T}_1 \to \mathcal{T}(n, G)$ which is a simplicial approximation to $f$. In particular, $d(h(x), f(x)) < \epsilon/3$ for all $x \in X$.

Consider the composite chain map

$$\mathcal{C}_p(K) \xrightarrow{\nu} \mathcal{C}_p(\mathcal{T}_1) \xrightarrow{h_*} \mathcal{C}_p(\mathcal{T}(n, G)) \xrightarrow{\mu} \mathcal{C}_p(K)$$

where $\nu$ denotes the subdivision chain map. Let $t$ denote the set-transformation [2] which associates to each open simplex $s$ of $K$ the closed subset $t(s) = \bigcup_{s \subseteq t} \pi_s \phi^{-1} h(|s|)$ of $X$. Then it is clear that $t$ is a carrier of the above composite chain map $\mu h \nu$.

We shall now show that $\bigcap t(s) = \emptyset$ for every simplex $s$ of $K$. Assume, by way of contradiction, that $x \in \bigcap t(s)$ for some $s$ of $K$. Then there exists $z \in X^n$ for which $\pi_s z = x$ for some $i$ and $z \in \phi^{-1} h(|s|)$. Since $z \in \phi^{-1} h(|s|)$, there exists $z' \in \eta^{-1} h(|s|)$ with $\phi(z') = z$. Hence $d(z, z') < \epsilon/3$ in $X^n$. Since $z' \in \eta^{-1} h(|s|)$, there exists $y \in |s|$ such that $h(y) = \eta(z')$ and since $x$ and $y \in |s|$, we have $d(x, y) < \epsilon/3$ in $X$. Therefore $\omega(y, \eta(z)) < \epsilon/3$, since $\pi_s z = x$ and $d(x, y) < \epsilon/3$. But

$$\omega(y, f(y)) \leq \omega(y, \eta(z)) + d(\eta(z), f(y))$$

$$\leq \omega(y, \eta(z)) + d(\eta(z), \eta(z')) + d(h(y), f(y))$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

This inequality contradicts the choice of $\epsilon$. Therefore $\bigcap t(s) = \emptyset$ for all simplexes $s$ of $K$. It follows that the chain mapping $\mu h^p \nu^p$ has, when expressed as a matrix, all zero entries on the diagonal. Therefore the trace of $\mu h^p \nu^p = 0$ for all $p$, and hence $\sum_{p=0}^{\infty} (-1)^p \text{trace}(\mu^p h^p \nu^p) = 0$. But the alternating sum of the traces of a chain map is equal to the alternating sum of the traces of its induced homology homomorphism [2]. Hence

$$\sum_{p=0}^{\infty} (-1)^p \text{trace}(\mu_* h_* \nu_*) = \sum_{p=0}^{\infty} (-1)^p \text{trace}(\mu^p h^p \nu^p) = \mathcal{L}(f) = 0$$
since $\nu^*_h$ is the identity and $h$ is homotopic to $f$. But this contradicts the hypothesis that $\mathcal{L}(f) \neq 0$. Therefore $f$ has a fixed point. Q.E.D.

**Corollary.** If $X$ is an acyclic polyhedron, then every function $f: X \to X^n/G$ has a fixed point.

**Proof.** Since $X$ is connected, the group $H_0(x)$ is one dimensional over the rationals. Let $z_0$ be any nonzero element of $H_0(X)$. It is straight-forward to check that $\mu^*_0 f^*_0(z_0) = nz_0$. Since $H_p(X) = 0$ for $p > 0$, we have $\mu^*_p f^*_p = 0$ for $p > 0$. Hence $\mathcal{L}(f) = n \neq 0$ and $f$ has a fixed point. Q.E.D.

Let $X$ be a connected polyhedron and let $x_0$ be an arbitrary point of $X$. For any integer $k$ where $1 \leq k \leq n$, define a map $d_k: X \to X^n$ to be the identity on the first $k$ factors and the constant value $x_0$ on the remaining $n - k$ factors. Let $\tilde{d}_k = \eta d_k$. Since $X$ is connected, the homotopy classes of $d_k$ and $\tilde{d}_k$ are independent of $x_0$. For all $k$, $\tilde{d}_k$ leaves every element of $X$ fixed.

**Corollary.** If the Euler characteristic $\chi(X) \neq (k - n)/k$, then every function $f: X \to X^n/G$ homotopic to $\tilde{d}_k$ has a fixed point.

**Proof.** Since $f$ is homotopic to $\tilde{d}_k$, we have $\mathcal{L}(f) = \mathcal{L}(\tilde{d}_k) = \mathcal{L}(\eta d_k)$. But $\mathcal{L}(\eta d_k)$ is the sum of the Lefschetz numbers of the components of $\tilde{d}_k$. Hence $\mathcal{L}(f) = k \chi(X) + n - k$. The hypothesis implies that $\mathcal{L}(f) \neq 0$. Therefore $f$ has a fixed point. Q.E.D.

5. Spaces with groups of operators. Coincidences. Let $Y$ be a space and $G$ a group of operators on $Y[1]$. Assume that $G$ is finite and let $g_1, \ldots, g_n$ be any enumeration of the elements of $G$. Let $Y/G$ be the space of orbits under $G$ in the identification topology and let $\pi: Y \to Y/G$ be the identification map.

Let $Y^\langle n \rangle = Y^n/S_n$ be the $n$th symmetric product of $Y$. Then the orbit space $Y/G$ can be embedded in $Y^\langle n \rangle$ as follows: Consider the map $\rho: Y \to Y^n$ whose components are the elements of $G$, that is $\rho(y) = (g_1y, \ldots, g_ny)$. For any $g \in G$, $\rho(gy) = (g_1gy, \ldots, g_ngy)$ which is equivalent to $\rho(y)$ in $Y^n$ under the symmetric group. Hence $\rho$ induces a single-valued function $\bar{\rho}: Y/G \to Y^\langle n \rangle$ which can be shown to be a homeomorphism, and the following diagram commutes:

\[
\begin{array}{ccc}
Y & \xrightarrow{\rho} & Y^n \\
\pi \downarrow & & \downarrow \eta \\
Y/G & \rightarrow & Y^\langle n \rangle
\end{array}
\]

Now let $f: Y \to Y/G$ be an arbitrary continuous function.
Theorem 2. If \( Y \) is a polyhedron and \( \mathcal{L}(\tilde{p}f) \neq 0 \), then \( f \) and \( \pi \) have a coincidence.

Proof. If \( \mathcal{L}(\tilde{p}f) \neq 0 \), then \( \tilde{p}f \) has a fixed point. Therefore there exists \( y \in Y \) which is a coordinate of \( \tilde{p}f(y) \). Let \( f(y) = x \in Y/G \) and let \( y' \) be any element of \( Y \) such that \( \pi(y') = x \). Then \( \tilde{p}f(y) = \tilde{p}(x) = (g_1 y', \ldots, g_n y') \). Hence \( y = g_1 y' \) for some \( g_1 \), and \( \pi(y) = \pi(y') = x = f(y) \). Q.E.D.

Corollary 1. Let \( Y \) and \( Y/G \) be polyhedra. If either \( Y \) or \( Y/G \) is acyclic, then any function \( f: Y \to Y/G \) has a coincidence with \( \pi \).

Proof. In each case \( \mathcal{L}(\tilde{p}f) \) is easily seen to be nonzero. Q.E.D.

Corollary 2. If \( Y \) is a polyhedron and \( \sum_{i=1}^{n} \mathcal{L}(g_i) \neq 0 \), then every function \( f: Y \to Y/G \) homotopic to \( \pi \) has a coincidence with \( \pi \).

Proof. If \( f \) is homotopic to \( \pi \), then \( \tilde{p}f \) is homotopic to \( \tilde{p} \pi \). But \( \tilde{p} \pi = \eta p \). Therefore \( \mathcal{L}(\tilde{p}f) = \mathcal{L}(\eta p) = \sum_{i=1}^{n} \mathcal{L}(g_i) \) since the elements of the group are the components of \( p \). By hypothesis \( \sum_{i=1}^{n} \mathcal{L}(g_i) \neq 0 \), so \( f \) has a coincidence with \( \pi \). Q.E.D.

Example. Let \( X \) be a \( k \)-sphere, and let \( G \) be any finite group of homeomorphisms on \( X \). Then if \( k \) is even, every function \( f: X \to X/G \) homotopic to \( \pi \) has a coincidence with \( \pi \). If \( k \) is odd and \( G \) has an element which reverses orientation, then every function \( f: X \to X/G \) homotopic to \( \pi \) has a coincidence with \( \pi \).

For, if we denote the degree of \( g_i \in G \) by \( a_i = \pm 1 \), we have

\[
\mathcal{L}(g_i) = 1 + (-1)^k a_i \geq 0.
\]

If \( k \) is even, \( \mathcal{L}(e) = 2 \) for the identity element \( e \) of the group and \( \sum_{i=1}^{n} \mathcal{L}(g_i) \neq 0 \). If \( k \) is odd and \( G \) has an orientation reversing element \( g_i \), then \( a_i = -1 \) and \( \mathcal{L}(g_i) = 2 \). Hence \( \sum_{i=1}^{n} \mathcal{L}(g_i) \neq 0 \).

Bibliography