FIXED POINTS OF SYMMETRIC PRODUCT MAPPINGS

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1. Introduction. Let X be a topological space and X^n the nth cartesian product in the usual topology. Let G be any group of permutations of the letters $[1, \cdots, n]$. Then G can be considered as a group of homeomorphisms on X^n by defining, for $g \in G$ and $(x_1, \cdots, x_n) \in X^n$, $g(x_1, \cdots, x_n) = (x_{g(1)}, \cdots, x_{g(n)})$. The orbit space under the group (in the identification topology) will be denoted by X^n/G and called a G-product of X. Let η: $X \to X^n/G$ be the identification map.[1]

We will consider continuous functions f: $X \to X^n/G$ and will say that an element $x \in X$ is *fixed* under f if $f(x)$ has x as one of its coordinates. The purpose of this paper is to associate to each G-product map a Lefschetz number (an integer depending upon the homotopy class of the function) with the property that if this number is not zero, then the function has a fixed point [2]. The Lefschetz number will be defined only in case X is a polyhedron. An application will be given regarding spaces with a finite group of operators.

2. G-products and a special homology homomorphism. Let the ith projection of X^n onto X be denoted by π_i. It is clear that the group and the projections are interrelated by $\pi_i g(z) = \pi_{g(i)} z$ for $z \in X^n$ and $g \in G$.

In case X is metric, then using the usual euclidean metric on X^n, G becomes a group of isometries on X^n. A metric may be introduced in X^n/G by defining

$$d(\eta(z), \eta(z')) = \inf \{d(z, gz') \mid g \in G\}$$

where $z, z' \in X^n$. It is convenient also to introduce a metric-like function on $X \times (X^n/G)$ defined by

$$\omega(x, \eta(z)) = \inf \{d(x, \pi_i(z)) \mid i = 1, \cdots, n\}$$

where $x \in X$ and $z \in X^n$. Then ω is continuous and satisfies the inequality:

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$$\omega(x, y) \leqq \omega(x, y') + d(y', y)$$

for any $x \in X$ and $y, y' \in X^n/G$.

Now let $X = |K|$ be a polyhedron. Denote the set of vertices of K by V and the set simplexes of K by S. Assume that K is an ordered complex, that is to say that a partial ordering $\leqq$ is defined on V which is a linear ordering on any subset of V in S. A triangulation K^n of X^n may be obtained as follows [1, p. 67]: The set of vertices of K^n is V^n. Let π_i: $V^n \to V$ be the ith projection. For $w, w' \in V^n$, define $w \leqq w'$ if $\pi_i w \leqq \pi_i w'$ for all $i = 1, \cdots, n$. A subset $t = (w_0, \cdots, w_p)$ of V^n is a simplex of K^n if it is linearly ordered and the vertices $\pi_i w_0, \cdots, \pi_i w_p$ span a simplex of K for $i = 1, \cdots, n$ (the vertices $\pi_i w_0, \cdots, \pi_i w_p$ need not be distinct). It follows from the definition of K^n that the projections are simplicial and that G is a group of order preserving functions on K^n. Since each $g \in G$ is order preserving on V^n, we have that if (w, gw) is a simplex of K^n, then $w \leqq gw \leqq g^2 w \leqq \cdots \leqq g^k w = w$ for some integer k, and hence $gw = w$.

Let $Sd(K^n)$ denote the first barycentric subdivision of K^n (the set B of vertices of $Sd(K^n)$ consists in all barycenters b_t of simplexes t of K^n). The group G operates on $Sd(K^n)$ by $gb_t = b_{gt}$ and the simplicial map ϕ: $Sd(K^n) \to K^n$ (which associates to b_t the least vertex of t) commutes with each $g \in G$. Furthermore, if $(b_t, b_{t'})$ and $(b_t, gb_{t'})$ are both simplexes of $Sd(K^n)$, then $gb_t = b_t$. For if $(b_t, b_{t'})$ and $(g^{-1}b_t, b_{t'})$ are both simplexes of $Sd(K^n)$ then t and $g^{-1}(t)$ are both faces of t'. If we let w denote any vertex of t, then $(w, g^{-1}w)$ is a simplex of K^n. By a previous argument, $w = gw$. Hence $t = gt$ and $gb_t = b_t$.

A triangulation $K(n, G)$ for X^n/G can now be defined. The set A of vertices of $K(n, G)$ is the set of equivalence class of elements of B under G. A subset $(a_0, \cdots, a_p)$ of A form a simplex of $K(n, G)$ if there exists $b_i \in a_i$ so that $(b_0, \cdots, b_p)$ is a simplex of $Sd(K^n)$. If another choice is made, say $b_i' \in a_i$ so that $(b_0', \cdots, b_p')$ is a simplex of $Sd(K^n)$, then $b_i' = g_i b_i$ for some $g_i \in G$, $i = 0 \cdots p$. For any i we have (b_i, b_p) and $(g_i b_i, g_p b_p)$ simplexes of $Sd(K^n)$, and therefore (b_i, b_p) and $(b_i, g_i^{-1} g_p b_p)$ are simplexes of $Sd(K^n)$. By a previous argument, we have $g_i^{-1} g_p b_i = b_i$, and hence $g_p b_i = g_i b_i$. Therefore

$$\begin{aligned}(b_0', \cdots, b_p') &= (g_0 b_0, \cdots, g_p b_p) = (g_p b_0, \cdots, g_p b_p) \\ &= g_p(b_0, \cdots, b_p).\end{aligned}$$

The p-simplexes of $K(n, G)$ are therefore in one-to-one correspondence with the equivalence classes of p-simplexes of $Sd(K^n)$.

Consider now the integral chain groups defined on oriented simplexes of these complexes:

$$\begin{array}{ccc} & C_p(K^n) \overset{\phi\#}{\leftarrow} C_p(Sd(K^n)) \\ \sum_{i=1}^{n} \pi_{i\#} \downarrow & & \downarrow \eta\# \\ & C_p(K) \qquad C_p(K(n, G)) \end{array}$$

We wish to complete the rectangle with a homomorphism μ: $C_p(K(n, G)) \rightarrow C_p(K)$ so that the diagram commutes ($\sum_{i=1}^{n} \pi_{i\#}$ is the sum of the projection chain maps). Let $t = (a_0, \cdots, a_p)$ be a generator of $C_p(K(n, G))$. Choose any $\sigma = (b_0, \cdots, b_p)$ generator of $C_p(Sd(K^n))$ for which $\eta\#(\sigma) = t$. Then define

$$\mu(t) = \sum_{i=1}^{n} \pi_{i\#}\phi\#(\sigma).$$

The definition is independent of the choice of σ. For if $\eta\#(\sigma') = \eta\#(\sigma) = t$, then $\sigma' = g\sigma$ for some $g \in G$, and

$$\sum_{i=1}^{n} \pi_{i\#}\phi\#(g\sigma) = \sum_{i=1}^{n} \pi_{i\#}g\#\phi\#(\sigma) = \sum_{i=1}^{n} \pi_{g(i)\#}\phi\#(\sigma)$$

$$= \sum_{i=1}^{n} \pi_{i\#}\phi\#(\sigma).$$

Also, μ is a chain map, since $\eta\#$, $\phi\#$, and $\sum_{i=1}^{n} \pi_{i\#}$ each commute with the boundary operator, and hence μ induces μ_*^d: $H_p(X^n/G) \rightarrow H_p(X)$.

From the commutativity of the above mapping diagram, we have $\sum_{i=1}^{n} \pi_{i*} = \mu_*\eta_*$ as mappings from $H_p(X^n)$ into $H_p(X)$. In general a G-product mapping $f: X \rightarrow X^n/G$ cannot be factored through X^n. However, if f can be expressed as a composite $\eta f'$ where $f': X \rightarrow X^n$, then

$$\mu_* f_* = \mu_*\eta_* f'_* = \sum_{i=1}^{n} \pi_{i*} f'_* = \sum_{i=1}^{n} f'_{i*}$$

where f'_i is the ith component $\pi_i f'$ of f'.

3. **The topological invariance of μ_*.** Let X and Y be any two spaces and $F: X \rightarrow Y$ a continuous function. Then F induces a map $\tilde{F}: X^n/G \rightarrow Y^n/G$ defined as follows: Let $F^n: X^n \rightarrow Y^n$ be given by $F^n(x_1, \cdots, x_n) = (Fx_1, \cdots, Fx_n)$. Then F^n is equivariant with respect to G and hence induces a continuous map $\tilde{F}$ on the quotient spaces. If $F_1: X \rightarrow Y$ and $F_2: Y \rightarrow Z$, then clearly the function induced on the corresponding G products by the composite F_2F_1 is the composite $\tilde{F}_2\tilde{F}_1$. Also, if F_0 and F_1 are two maps of X into Y which are homotopic, then $\tilde{F}_0$ and $\tilde{F}_1$ are homotopic. If $F: X \rightarrow Y$ is a homeomorphism, then $\tilde{F}$ is also.

Let $X=|K|$ and $Y=|L|$ be polyhedra where K and L are ordered complexes. Then the complexes $K(n, G)$ and $L(n, G)$ are defined as before and depend upon the vertex ordering of K and L, respectively. Let $f: K\rightarrow L$ be a simplicial order preserving function. Then f^n can be defined on the vertices of K^n into the vertices of L^n by $f^n(v_1\times \cdots \times v_n)=(fv_1\times \cdots \times fv_n)$ and f^n is order preserving on the vertices of K^n. So, if $w_0\leqq \cdots \leqq w_p$ is a simply ordered set of vertices of K^n, $f^n w_0\leqq \cdots \leqq f^n w_p$ is simply ordered. Furthermore, $\bar{\pi}_i f^n=f\pi_i$ for all $i=1, \cdots, n$ where π_i and $\bar{\pi}_i$ are the projections in K and L, respectively. Hence, $f^n: K^n\rightarrow L^n$ is a simplicial map and induces $(f^n)': Sd(K^n)\rightarrow Sd(L^n)$ in the usual fashion. The map f^n is equivariant with respect to G and hence $(f^n)'$ is also. Therefore, $(f^n)'$ induces a simplicial function $\bar{f}: K(n, G)\rightarrow L(n, G)$, and the continuous function induces on $|K|^n/G$ into $|L|^n/G$ by f is the same as the simplicial map $\bar{f}$. If we denote by $\phi: Sd(K^n)\rightarrow K^n$ and $\bar{\phi}: Sd(L^n)\rightarrow L^n$ the order preserving subdivision maps, then we have $f^n\phi=\bar{\phi}(f^n)'$. If $\eta: Sd(K^n)\rightarrow K(n, G)$ and $\bar{\eta}: Sd(L^n)\rightarrow L(n, G)$ are the identification maps, then $\bar{f}\eta=\bar{\eta}(f^n)'$.

LEMMA 1. *If $f: K\rightarrow L$ is an order preserving simplicial map, then $f_*\mu_{K*}=\mu_{L*}\bar{f}_*$.*

PROOF. Take σ a generator of $C_p(K(n, G))$ and let t be a generator of $C_p(Sd(K^n))$ for which $\eta(t)=\sigma$. Then

$$
\begin{aligned}
f_\#\mu_K(\sigma) &= f_\# \sum_{i=1}^{n} \pi_{i\#}\phi_\#(t) \\
&= \sum_{i=1}^{n} \bar{\pi}_{i\#}f^n_\# \phi_\#(t) \\
&= \sum_{i=1}^{n} \bar{\pi}_{i\#}\bar{\phi}_\#(f^n)'_\# (t) \\
&= \mu_L(\bar{f}_\#\sigma)
\end{aligned}
$$

since $\bar{\eta}((f^n)'(t))=\bar{f}\eta(t)=\bar{f}(\sigma)$. Q.E.D.

LEMMA 2. *If $f:K\rightarrow L$ is any simplicial function (not necessarily order preserving), then $f_*\mu_{K*}=\mu_{L*}\bar{f}_*$.*

PROOF. Let K' and L' be the barycentric subdivision of K and L, respectively. Let $\alpha: K'\rightarrow K$ and $\beta: L'\rightarrow L$ be the simplicial maps which associate to each barycenter of a simplex the least vertex of the simplex. Then α and β are order preserving. Let $f': K'\rightarrow L'$ be the simplicial map induced by f. Then f' is order preserving, and we have

$$\begin{array}{ccccccc} H_p(K) & \xleftarrow{\alpha_*} & H_p(K') & \xrightarrow{f'_*} & H_p(L') & \xrightarrow{\beta_*} & H_p(L) \\ \uparrow \mu_{K*} & & \uparrow \mu_{K'*} & & \uparrow \mu_{L'*} & & \uparrow \mu_{L*} \\ H_p(K(n,G)) & \xleftarrow{\tilde{\alpha}_*} & H_p(K'(n,G)) & \xrightarrow{\tilde{f}'} & H_p(L'(n,G)) & \xrightarrow{\tilde{\beta}_*} & H_p(L(n,G)). \end{array}$$

By Lemma 1, each rectangle commutes. It is clear that α_* is an isomorphism and that $f_* = \beta_* f'_* \alpha_*^{-1}$. Since α and β are homotopic to the identity on $|K|$ and $|L|$, respectively, $\tilde{\alpha}$ and $\tilde{\beta}$ are homotopic to the identity on $|K(n, G)|$ and $|L(n, G)|$. Therefore, $\tilde{\alpha}_*$ and $\tilde{\beta}_*$ are isomorphisms. Since $f\alpha$ and $\beta f'$ are homotopic, we have $\tilde{f}\tilde{\alpha}$ homotopic to $\tilde{\beta}\tilde{f}'$. Hence, $\tilde{f}_*\tilde{\alpha}_* = \tilde{\beta}_*\tilde{f}'_*$ and $\tilde{f}_* = \tilde{\beta}_*\tilde{f}'_*\tilde{\alpha}_*^{-1}$. Therefore, $f_*\mu_{K*} = \mu_{L*}\tilde{f}_*$.

LEMMA 3. *If F: $|K| \to |L|$ is any continuous function, then* $F_*\mu_{K*} = \mu_{L*}\tilde{F}_*$.

PROOF. Let N be a barycentric subdivision of K, sufficiently fine so that a simplicial map f: $N \to L$ exists which is a simplicial approximation to F[2]. Let γ: $N \to K$ be a simplicial subdivision map. We have the diagram:

$$\begin{array}{ccccc} H_p(K) & \xleftarrow{\gamma_*} & H_p(N) & \xrightarrow{f_*} & H_p(L) \\ \uparrow \mu_{K*} & & \uparrow \mu_{N*} & & \uparrow \mu_{L*} \\ H_p(K(n,G)) & \xleftarrow{\tilde{\gamma}_*} & H_p(N(n,G)) & \xrightarrow{\tilde{f}_*} & H_p(L(n,G)) \end{array}$$

By the previous lemma, each rectangle commutes. In the top row, γ_* is an isomorphism and $F_* = f_*\gamma_*^{-1}$. Since γ is homotopic to the identity on $|K|$, $\tilde{\gamma}$ is homotopic to the identity on $|K|^n/G$ and $\tilde{\gamma}_*$ is an isomorphism. Since $F\gamma$ is homotopic to F, we have $\tilde{F}\tilde{\gamma}$ homotopic to $\tilde{F}$. Hence $\tilde{F}_*\tilde{\gamma}_* = \tilde{f}_*$ and $\tilde{F}_* = \tilde{f}_*\tilde{\gamma}_*^{-1}$. Therefore, $F_*\mu_{K*} = \mu_{L*}\tilde{F}_*$.

Now if we take F to be a homeomorphism, we have:

COROLLARY. *The homomorphism μ_* is independent of the triangulation.*

4. **Main theorem.** As before, let $X = |K|$ be a polyhedron and let f: $X \to X^n/G$ be a continuous function. In the remainder of the paper the coefficient group for the homology groups will be the field of rational numbers. Consider the composite homomorphism

$$H_p(X) \xrightarrow{f_*^p} H_p(X^n/G) \xrightarrow{\mu_*^p} H_p(X)$$

Since $\mu_*^p f_*^p$ is a linear transformation of a finite dimensional vector space into itself, the trace of $\mu_*^p f_*^p$ is defined. The Lefschetz number $\mathfrak{L}(f)$ is defined to be the number $\sum_{p=0}^{\infty} (-1)^p$ trace $(\mu_*^p f_*^p)$. Since f_*^p

and μ_*^p are independent of the triangulation, $\mathcal{L}(f)$ is also. In case $n=1$, then μ_*^p is the identity and $\mathcal{L}(f)$ reduces to the Lefschetz number as defined in [2].

THEOREM 1. *If X is a polyhedron and $f: X \to X^n/G$ where $\mathcal{L}(f) \neq 0$, then f has a fixed point.*

PROOF. Assume, by way of contradiction, that f has no fixed points. Then $\omega(x, f(x)) > 0$ for all $x \in X$. Since X is compact, there exists $\epsilon > 0$ such that $\omega(x, f(x)) > \epsilon$ for all $x \in X$.

Let a triangulation K of X be chosen so small that mesh $K^n < \epsilon/3$. Then clearly mesh $K < \epsilon/3$ and mesh $K(n, G) < \epsilon/3$. By the simplicial approximation theorem, there is a subdivision K_1 of K and a map $h: K_1 \to K(n, G)$ which is a simplicial approximation to f. In particular, $d(h(x), f(x)) < \epsilon/3$ for all $x \in X$.

Consider the composite chain map

$$C_p\,(K) \xrightarrow{\nu} C_p(K_1) \xrightarrow{h_\#} C_p(K(n,G)) \xrightarrow{\mu} C_p(K)$$

where ν denotes the subdivision chain map. Let t denote the set-transformation [2] which associates to each open simplex s of K the closed subset $t(s) = \bigcup_{i=1}^n \pi_i \phi \eta^{-1} h(|s|)$ of X. Then it is clear that t is a carrier of the above composite chain map $\mu h_\# \nu$.

We shall now show that $|s| \cap t(s) = \phi$ for every simplex s of K. Assume, by way of contradiction, that $x \in |s| \cap t(s)$ for some s of K. Then there exists $z \in X^n$ for which $\pi_i z = x$ for some i and $z \in \phi\eta^{-1}h(|s|)$. Since $z \in \phi\eta^{-1}h(|s|)$, there exists $z' \in \eta^{-1}h(|s|)$ with $\phi(z') = z$. Hence $d(z, z') < \epsilon/3$ in X^n. Since $z' \in \eta^{-1}h(|s|)$, there exists $y \in |s|$ such that $h(y) = \eta(z')$ and since x and $y \in |s|$, we have $d(x, y) < \epsilon/3$ in X. Therefore $\omega(y, \eta(z)) < \epsilon/3$, since $\pi_i z = x$ and $d(x, y) < \epsilon/3$. But

$$\begin{aligned} \omega(y, f(y)) &\leqq \omega(y, \eta(z)) + d(\eta(z), f(y)) \\ &\leqq \omega(y, \eta(z)) + d(\eta(z), \eta(z')) + d(h(y), f(y)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

This inequality contradicts the choice of ϵ. Therefore $|s| \cap t(s) = \phi$ for all simplexes s of K. It follows that the chain mapping $\mu^p h_\#^p \nu^p$ has, when expressed as a matrix, all zero entries on the diagonal. Therefore the trace of $\mu^p h_\#^p \nu^p = 0$ for all p, and hence $\sum_{p=0}^\infty (-1)^p \operatorname{trace}(\mu^p h_\#^p \nu^p) = 0$. But the alternating sum of the traces of a chain map is equal to the alternating sum of the traces of its induced homology homomorphism [2]. Hence

$$\sum_{p=0}^\infty (-1)^p \operatorname{trace}\,(\mu_*^p h_*^p \nu_*^p) = \sum_{p=0}^\infty (-1)^p \operatorname{trace}\,(\mu_*^p f_*^p) = \mathcal{L}(f) = 0$$

since ν_*^p is the identity and h is homotopic to f. But this contradicts the hypothesis that $\mathcal{L}(f) \neq 0$. Therefore f has a fixed point. Q.E.D.

COROLLARY. *If X is an acyclic polyhedron, then every function $f: X \rightarrow X^n/G$ has a fixed point.*

PROOF. Since X is connected, the group $H_0(x)$ is one dimensional over the rationals. Let z_0 be any nonzero element of $H_0(X)$. It is straight-forward to check that $\mu_*^0 f_*^0(z_0) = nz_0$. Since $H_p(X) = 0$ for $p > 0$, we have $\mu_*^p f_*^p = 0$ for $p > 0$. Hence $\mathcal{L}(f) = n \neq 0$ and f has a fixed point. Q.E.D.

Let X be a connected polyhedron and let x_0 be an arbitrary point of X. For any integer k where $1 \leqq k \leqq n$, define a map $d_k: X \rightarrow X^n$ to be the identity on the first k factors and the constant value x_0 on the remaining $n-k$ factors. Let $\bar{d}_k = \eta d_k$. Since X is connected, the homotopy classes of d_k and $\bar{d}_k$ are independent of x_0. For all k, $\bar{d}_k$ leaves every element of X fixed.

COROLLARY. *If the Euler characteristic $\mathfrak{X}(X) \neq (k-n)/k$, then every function $f: X \rightarrow X^n/G$ homotopic to $\bar{d}_k$ has a fixed point.*

PROOF. Since f is homotopic to $\bar{d}_k$, we have $\mathcal{L}(f) = \mathcal{L}(\bar{d}_k) = \mathcal{L}(\eta d_k)$. But $\mathcal{L}(\eta d_k)$ is the sum of the Lefschetz numbers of the components of d_k. Hence $\mathcal{L}(f) = k\mathfrak{X}(X) + n - k$. The hypothesis implies that $\mathcal{L}(f) \neq 0$. Therefore f has a fixed point. Q.E.D.

5. **Spaces with groups of operators. Coincidences.** Let Y be a space and G a group of operators on Y[1]. Assume that G is finite and let $g_1, \cdots, g_n$ be any enumeration of the elements of G. Let Y/G be the space of orbits under G in the identification topology and let $\pi: Y \rightarrow Y/G$ be the identification map.

Let $Y^{(n)} = Y^n/S_n$ be the nth symmetric product of Y. Then the orbit space Y/G can be embedded in $Y^{(n)}$ as follows: Consider the map $\rho: Y \rightarrow Y^n$ whose components are the elements of G, that is $\rho(y) = (g_1 y, \cdots, g_n y)$. For any $g \in G$, $\rho(gy) = (g_1 gy, \cdots, g_n gy)$ which is equivalent to $\rho(y)$ in Y^n under the symmetric group. Hence ρ induces a single-valued function $\bar{\rho}: Y/G \rightarrow Y^{(n)}$ which can be shown to be a homeomorphism, and the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & Y^n \\ \pi \downarrow & & \downarrow \eta \\ Y/G & \xrightarrow{\bar{\rho}} & Y^{(n)} \end{array}$$

Now let $f: Y \rightarrow Y/G$ be an arbitrary continuous function.

THEOREM 2. *If Y is a polyhedron and $\mathfrak{L}(\bar{\rho}f)\neq 0$, then f and π have a coincidence.*

PROOF. If $\mathfrak{L}(\bar{\rho}f)\neq 0$, then $\bar{\rho}f$ has a fixed point. Therefore there exists $y\in Y$ which is a coordinate of $\bar{\rho}f(y)$. Let $f(y)=x\in Y/G$ and let y' be any element of Y such that $\pi(y')=x$. Then $\bar{\rho}f(y)=\bar{\rho}(x)=(g_1y',\cdots,g_ny')$. Hence $y=g_iy'$ for some g_i, and $\pi(y)=\pi(y')=x=f(y)$. Q.E.D.

COROLLARY 1. *Let Y and Y/G be polyhedra. If either Y or Y/G is acyclic, then any function f: $Y\rightarrow Y/G$ has a coincidence with π.*

PROOF. In each case $\mathfrak{L}(\bar{\rho}f)$ is easily seen to be nonzero. Q.E.D.

COROLLARY 2. *If Y is a polyhedron and $\sum_{i=1}^{n}\mathfrak{L}(g_i)\neq 0$, then every function f: $Y\rightarrow Y/G$ homotopic to π has a coincidence with π.*

PROOF. If f is homotopic to π, then $\bar{\rho}f$ is homotopic to $\bar{\rho}\pi$. But $\bar{\rho}\pi=\eta\rho$. Therefore $\mathfrak{L}(\bar{\rho}f)=\mathfrak{L}(\eta\rho)=\sum_{i=1}^{n}\mathfrak{L}(g_i)$ since the elements of the group are the components of ρ. By hypothesis $\sum_{i=1}^{n}\mathfrak{L}(g_i)\neq 0$, so f has a coincidence with π. Q.E.D.

EXAMPLE. Let X be a k-sphere, and let G be any finite group of homeomorphisms on X. Then if k is even, every function f: $X\rightarrow X/G$ homotopic to π has a coincidence with π. If k is odd and G has an element which reverses orientation, then every function f: $X\rightarrow X/G$ homotopic to π has a coincidence with π.

For, if we denote the degree of $g_i\in G$ by $a_i=\pm 1$, we have

$$\mathfrak{L}(g_i) = 1 + (-1)^k a_i \geqq 0.$$

If k is even, $\mathfrak{L}(e)=2$ for the identity element e of the group and $\sum_{i=1}^{n}\mathfrak{L}(g_i)\neq 0$. If k is odd and G has an orientation reversing element g_j, then $a_j=-1$ and $\mathfrak{L}(g_j)=2$. Hence $\sum_{i=1}^{n}\mathfrak{L}(g_i)\neq 0$.

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