A NOTE ON THE SPAN OF TRANSLATIONS IN L^p

C. S. HERZ[1]

Suppose $f \in L^1 \cap L^p$. f is said to have the Wiener closure property,[2] (C), if the translates of f span L^p. Since $f \in L^1$, the Fourier transform $\hat{f}$ is well defined. Let $Z(f)$ be the set of zeros of $\hat{f}$. One would like to reformulate (C) in terms of structural properties of the closed set $Z(f)$. The problem seems quite difficult; in this note we show that (C) is nearly equivalent to a uniqueness property of $Z(f)$.[3]

It is assumed that the notion of the spectrum[4] of a bounded continuous function is familiar.

DEFINITION. *A closed set is of type U^q if the only bounded continuous function in L^q with spectrum contained in the set is the null function.*[5]

We shall say that f has property (U) if $Z(f)$ is of type U^q where $1/p+1/q=1$. Pollard, [4], has observed, what is true for any locally compact Abelian group, that

THEOREM 1. *For $1 \leqq p < \infty$, (U) implies (C).*

In the converse direction one has trivially,

THEOREM 2. *For $2 \leqq p < \infty$, (C) implies (U).*

Of course, this also holds for $p=1$. What is left open is the case $1<p<2$. Here we have two classes of results corresponding to weakening the conclusion and strengthening the hypothesis respectively.

DEFINITION. *A closed set is of type U^{q*} if there is no nontrivial complex measure of bounded variation with spectrum (support) in the set whose Fourier-Stieltjes transform belongs to L^q.*

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[2] For a general discussion of the problem see [6] of the bibliography. The reference is to Part III, §2.

[3] This is the viewpoint of [4].

[4] For an elaborate treatment of spectral theory see [3]; however [5] will be more accessible to the classical analyst. For assertions about the spectrum not proved in the text see these references.

[5] The definition here is equivalent to that in [4]; the proof of equivalence is essentially the same as the proofs given there. Simple modifications of the method show that sets of uniqueness can be defined using any of a large variety of summability methods, including, when $q<\infty$, ordinary convergence of trigonometric integrals.

We shall say that f has property (U^*) if $Z(f)$ is of type U^{q*} where $1/p+1/q=1$.

THEOREM 2*. *For* $1\leqq p<\infty$, *(C) implies* (U^*).

The only result which requires any imagination is the next. It should be noted that the proof makes only trivial use of the "natural" assumption, $f\in L^p$, but it depends strongly on the fact that $f\in L^1$.

THEOREM 3. *If for some* $\epsilon>0$, $\hat{f}\in \mathrm{Lip}\,\epsilon$, *then (C) implies (U).*

We remark that the extra hypothesis is certainly fulfilled[6] if $\int|f(x)|\,|x|^\epsilon dx<\infty$.

To prove the above theorems, first observe that (C) is equivalent to the statement: if $\phi\in L^q$ and the convolution $f*\phi=0$, then $\phi=0$. Let $g\in L^1$ be such that $\hat{g}$ vanishes outside a compact set. If $f*\phi=0$ then $f*(g*\phi)=0$ while on the other hand, $\phi=0$ if and only if $g*\phi=0$ for each such g. Thus we may replace ϕ if necessary by $g*\phi$ and consider only bounded continuous functions $\phi\in L^q$ with compact spectrum $\Lambda(\phi)$. The defining property of the spectrum is that $f*\phi=0$ implies $\Lambda(\phi)\subset Z(f)$; this proves Theorem 1. The propositions in the converse direction are argued by contradiction. We assume there exists some non-null $\phi\in L^q$ with $\Lambda(\phi)\subset Z(f)$ and wish to prove that $f*\phi=0$, or something just as good. This is essentially a spectral synthesis problem, and as such it appears to require extra conditions. For example if ϕ is known to be a Fourier-Stieltjes transform, $\Lambda(\phi)\subset Z(f)$ implies $f*\phi=0$; this establishes Theorem 2*. The observation that it suffices to consider ϕ's with compact spectrum shows that for $1\leqq q\leqq 2$, type U^q is identical with type U^{q*} since every $\phi\in L^q$ with compact spectrum is a Fourier transform. Theorem 2 is therefore an immediate corollary of Theorem 2*.

All the foregoing is valid for locally compact Abelian groups. However, for simplicity, we present the details of the proof of Theorem 3 only for the real line. The extension to the general case is clearly indicated in [3], (cf. the proof there of Lemma 4.4). Suppose $Z(f)$ is not of type U^q. Then there is a non-null $\phi\in L^q$ with compact spectrum $\Lambda(\phi)\subset Z(f)$. Let $f^{(n)}$ denote the convolution of f with itself n times. If we can show that $f^{(n)}*\phi=0$ for some n we are through, for let n be the first integer for which this is true. If $n=1$, fine! Otherwise $f^{(n-1)}*\phi$ is a non-null function in L^q with spectrum $\Lambda(f^{(n-1)}*\phi)\subset\Lambda(\phi)\subset Z(f)$ and $f*(f^{(n-1)}*\phi)=0$. The Lipschitz condition is just what we need to guarantee the existence of an n so that $f^{(n)}*\phi=0$.

Choose an $h>0$ and set $k(x)=(x/2)^{-2}\sin^2 x/2$. Define $\Phi_h(t)$

[6] Theorem 3 is supposed to compare favorably with Theorem B of [4].

$=(2\pi)^{-1}\int \exp(-itx)k(hx)\phi(x)dx$. Then Φ_h vanishes outside the set Λ^h consisting of those points at a distance $<h$ from $\Lambda(\phi)$. Moreover $f^{(n)}*\phi(x) = \int f^{(n)}(x-y)\phi(y)dy = \lim_{h\to 0}\int f^{(n)}(x-y)k(hy)\phi(y)dy$ $=\lim_{h\to 0}\int \exp(itx)\hat{f}^n(t)\Phi_h(t)dt$. Hence it suffices to prove that $\int|\hat{f}^n(t)\Phi_h(t)|dt=o(1)$ as $h\to 0$. Now $\hat{f}$ vanishes on $\Lambda(\phi)$ and $\hat{f}\in \mathrm{Lip}\ \epsilon$. Hence if t is within h of $\Lambda(\phi)$, i.e., $t\in\Lambda^h$, $\hat{f}(t)=O(h^\epsilon)$. Since the integration is extended only over Λ^h, $\int|\hat{f}^n(t)\Phi_h(t)|dt=O(h^{n\epsilon})\int|\Phi_h(t)|dt$. The last integral obviously is $O(h^{-\delta})$ for some δ (a careful estimate will be considered later) so choose $n>\delta/\epsilon$.

The question of the structure of sets of type U^q is quite open. Let T be a closed set and $|T|$ its Lebesgue measure. Obviously $|T|=0$ is, in case $q\leqq 2$ a sufficient, and in case $q\geqq 2$ a necessary condition that T be of type U^q. Exact criteria are available for $q=1$ (T has empty interior), $q=2(|T|=0)$, and $q=\infty$ (T is empty). One would like to interpolate. The next theorem is a step in that direction which gives some content to Theorem 1. We consider r-tuple trigonometric series or integrals. Λ^h has the same meaning as in the paragraph above, and dim T is the Hausdorff dimension of T.

THEOREM 4. *Alternative sufficient conditions that the closed set T be of type U^q, $q\geqq 2$ are*

(i) $|\Lambda^h|=o(h^{r(1-2/q)})$ *for each compact subset* Λ *of* T,

(ii) dim $T<2r/q$, *with the proviso, if* $r>2$, *that* $q\leqq 2r/(r-2)$.

We shall give the proof of (i) only for ordinary trigonometric integrals. It suffices to show that if $\phi\in L^q$ is a bounded continuous function with compact spectrum $\Lambda(\phi)\subset T$ then $\phi=0$. This will be true if, in the previous notation, $\int|\Phi_h(t)|dt=o(1)$ as $h\to 0$. Using the Schwarz inequality, $\{\int|\Phi_h(t)|dt\}^2\leqq|\Lambda^h|\cdot\int|\Phi_h(t)|^2dt$. Next we employ the Parseval relation and the Hölder inequality.

$$\begin{aligned}\int|\Phi_h(t)|^2dt &= \int|k(hx)\phi(x)|^2dx\\ &\leqq \left\{\int|k(hx)^{2q/(q-2)}dx\right\}^{1-2/q}\cdot\left\{\int|\phi(x)|^qdx\right\}^{2/q}\\ &= O(h^{-1+2/q})\cdot O(1).\end{aligned}$$

Combining the estimates, $\{\int|\Phi_h(t)|dt\}^2=|\Lambda^h|\cdot O(h^{-1+2/q})=o(1)$ since $|\Lambda^h|=o(h^{1-2/q})$ by hypothesis. (ii) was proved by Beurling [1] for $r=1$ and extended by Deny [2, pp. 144–145].

The conditions of Theorem 4 are clearly unnecessary since an ordinary set of uniqueness is of type U^q for every q, $1\leqq q<\infty$. However the estimates cannot be improved.

In conclusion we mention one amusing problem for r-dimensional Euclidean space. Suppose $f \in L^p$ and vanishes outside a compact set. Then $\hat{f}$ is an entire function of exponential type. For $r=1$, the translates of f span L^p for all p, $1<p<\infty$, since $Z(f)$ is countable. However consideration of a few Bessel functions leads to the conclusion that for $r>1$ the theorem is certainly false unless $p \geqq 2r/(r+1)$. Is this a sufficient condition? Posing the problem otherwise, for what q is the set of real zeros of an entire function of exponential type in r-variables necessarily of type U^q?

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Cornell University