# A NOTE ON THE SPAN OF TRANSLATIONS IN $L^p$

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Suppose $f \in L^1 \cap L^p$. $f$ is said to have the Wiener closure property,[2] $(C)$, if the translates of $f$ span $L^p$. Since $f \in L^1$, the Fourier transform $\hat{f}$ is well defined. Let $Z(f)$ be the set of zeros of $\hat{f}$. One would like to reformulate $(C)$ in terms of structural properties of the closed set $Z(f)$. The problem seems quite difficult; in this note we show that $(C)$ is nearly equivalent to a uniqueness property of $Z(f)$.[3]

It is assumed that the notion of the spectrum[4] of a bounded continuous function is familiar.

DEFINITION. *A closed set is of type $U^q$ if the only bounded continuous function in $L^q$ with spectrum contained in the set is the null function.*[5]

We shall say that $f$ has property $(U)$ if $Z(f)$ is of type $U^q$ where $1/p+1/q=1$. Pollard, [4], has observed, what is true for any locally compact Abelian group, that

THEOREM 1. *For $1 \leqq p < \infty$, $(U)$ implies $(C)$.*

In the converse direction one has trivially,

THEOREM 2. *For $2 \leqq p < \infty$, $(C)$ implies $(U)$.*

Of course, this also holds for $p=1$. What is left open is the case $1<p<2$. Here we have two classes of results corresponding to weakening the conclusion and strengthening the hypothesis respectively.

DEFINITION. *A closed set is of type $U^{q*}$ if there is no nontrivial complex measure of bounded variation with spectrum (support) in the set whose Fourier-Stieltjes transform belongs to $L^q$.*

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[2] For a general discussion of the problem see [6] of the bibliography. The reference is to Part III, §2.

[3] This is the viewpoint of [4].

[4] For an elaborate treatment of spectral theory see [3]; however [5] will be more accessible to the classical analyst. For assertions about the spectrum not proved in the text see these references.

[5] The definition here is equivalent to that in [4]; the proof of equivalence is essentially the same as the proofs given there. Simple modifications of the method show that sets of uniqueness can be defined using any of a large variety of summability methods, including, when $q < \infty$, ordinary convergence of trigonometric integrals.

We shall say that $f$ has property $(U^*)$ if $Z(f)$ is of type $U^{q*}$ where $1/p+1/q=1$.

THEOREM 2*. *For* $1 \leqq p < \infty$, $(C)$ *implies* $(U^*)$.

The only result which requires any imagination is the next. It should be noted that the proof makes only trivial use of the "natural" assumption, $f \in L^p$, but it depends strongly on the fact that $f \in L^1$.

THEOREM 3. *If for some* $\epsilon > 0$, $\hat{f} \in \text{Lip } \epsilon$, *then* $(C)$ *implies* $(U)$.

We remark that the extra hypothesis is certainly fulfilled[6] if $\int |f(x)| \, |x|^\epsilon dx < \infty$.

To prove the above theorems, first observe that $(C)$ is equivalent to the statement: if $\phi \in L^q$ and the convolution $f * \phi = 0$, then $\phi = 0$. Let $g \in L^1$ be such that $\hat{g}$ vanishes outside a compact set. If $f * \phi = 0$ then $f * (g * \phi) = 0$ while on the other hand, $\phi = 0$ if and only if $g * \phi = 0$ for each such $g$. Thus we may replace $\phi$ if necessary by $g * \phi$ and consider only bounded continuous functions $\phi \in L^q$ with compact spectrum $\Lambda(\phi)$. The defining property of the spectrum is that $f * \phi = 0$ implies $\Lambda(\phi) \subset Z(f)$; this proves Theorem 1. The propositions in the converse direction are argued by contradiction. We assume there exists some non-null $\phi \in L^q$ with $\Lambda(\phi) \subset Z(f)$ and wish to prove that $f * \phi = 0$, or something just as good. This is essentially a spectral synthesis problem, and as such it appears to require extra conditions. For example if $\phi$ is known to be a Fourier-Stieltjes transform, $\Lambda(\phi) \subset Z(f)$ implies $f * \phi = 0$; this establishes Theorem 2*. The observation that it suffices to consider $\phi$'s with compact spectrum shows that for $1 \leqq q \leqq 2$, type $U^q$ is identical with type $U^{q*}$ since every $\phi \in L^q$ with compact spectrum is a Fourier transform. Theorem 2 is therefore an immediate corollary of Theorem 2*.

All the foregoing is valid for locally compact Abelian groups. However, for simplicity, we present the details of the proof of Theorem 3 only for the real line. The extension to the general case is clearly indicated in [3], (cf. the proof there of Lemma 4.4). Suppose $Z(f)$ is not of type $U^q$. Then there is a non-null $\phi \in L^q$ with compact spectrum $\Lambda(\phi) \subset Z(f)$. Let $f^{(n)}$ denote the convolution of $f$ with itself $n$ times. If we can show that $f^{(n)} * \phi = 0$ for some $n$ we are through, for let $n$ be the first integer for which this is true. If $n=1$, fine! Otherwise $f^{(n-1)} * \phi$ is a non-null function in $L^q$ with spectrum $\Lambda(f^{(n-1)} * \phi) \subset \Lambda(\phi) \subset Z(f)$ and $f * (f^{(n-1)} * \phi) = 0$. The Lipschitz condition is just what we need to guarantee the existence of an $n$ so that $f^{(n)} * \phi = 0$.

Choose an $h > 0$ and set $k(x) = (x/2)^{-2} \sin^2 x/2$. Define $\Phi_h(t)$

[6] Theorem 3 is supposed to compare favorably with Theorem B of [4].

$=(2\pi)^{-1}\int \exp(-itx)k(hx)\phi(x)dx$. Then $\Phi_h$ vanishes outside the set $\Lambda^h$ consisting of those points at a distance $<h$ from $\Lambda(\phi)$. Moreover $f^{(n)}*\phi(x) = \int f^{(n)}(x-y)\phi(y)dy = \lim_{h\to 0}\int f^{(n)}(x-y)k(hy)\phi(y)dy$ $=\lim_{h\to 0}\int \exp(itx)\hat{f}^n(t)\Phi_h(t)dt$. Hence it suffices to prove that $\int|\hat{f}^n(t)\Phi_h(t)|dt=o(1)$ as $h\to 0$. Now $\hat{f}$ vanishes on $\Lambda(\phi)$ and $\hat{f}\in\text{Lip }\epsilon$. Hence if $t$ is within $h$ of $\Lambda(\phi)$, i.e., $t\in\Lambda^h$, $\hat{f}(t)=O(h^\epsilon)$. Since the integration is extended only over $\Lambda^h$, $\int|\hat{f}^n(t)\Phi_h(t)|dt=O(h^{n\epsilon})\int|\Phi_h(t)|dt$. The last integral obviously is $O(h^{-\delta})$ for some $\delta$ (a careful estimate will be considered later) so choose $n>\delta/\epsilon$.

The question of the structure of sets of type $U^q$ is quite open. Let $T$ be a closed set and $|T|$ its Lebesgue measure. Obviously $|T|=0$ is, in case $q\leqq 2$ a sufficient, and in case $q\geqq 2$ a necessary condition that $T$ be of type $U^q$. Exact criteria are available for $q=1$ ($T$ has empty interior), $q=2$ ($|T|=0$), and $q=\infty$ ($T$ is empty). One would like to interpolate. The next theorem is a step in that direction which gives some content to Theorem 1. We consider $r$-tuple trigonometric series or integrals. $\Lambda^h$ has the same meaning as in the paragraph above, and dim $T$ is the Hausdorff dimension of $T$.

THEOREM 4. *Alternative sufficient conditions that the closed set $T$ be of type $U^q$, $q\geqq 2$ are*

(i) $|\Lambda^h|=o(h^{r(1-2/q)})$ *for each compact subset $\Lambda$ of $T$,*

(ii) dim $T<2r/q$, *with the proviso, if $r>2$, that $q\leqq 2r/(r-2)$.*

We shall give the proof of (i) only for ordinary trigonometric integrals. It suffices to show that if $\phi\in L^q$ is a bounded continuous function with compact spectrum $\Lambda(\phi)\subset T$ then $\phi=0$. This will be true if, in the previous notation, $\int|\Phi_h(t)|dt=o(1)$ as $h\to 0$. Using the Schwarz inequality, $\{\int|\Phi_h(t)|dt\}^2\leqq|\Lambda^h|\cdot\int|\Phi_h(t)|^2dt$. Next we employ the Parseval relation and the Hölder inequality.

$$\begin{aligned}\int|\Phi_h(t)|^2dt &= \int|k(hx)\phi(x)|^2dx\\ &\leqq\left\{\int|k(hx)^{2q/(q-2)}dx\right\}^{1-2/q}\cdot\left\{\int|\phi(x)|^q dx\right\}^{2/q}\\ &= O(h^{-1+2/q})\cdot O(1).\end{aligned}$$

Combining the estimates, $\{\int|\Phi_h(t)|dt\}^2=|\Lambda^h|\cdot O(h^{-1+2/q})=o(1)$ since $|\Lambda^h|=o(h^{1-2/q})$ by hypothesis. (ii) was proved by Beurling [**1**] for $r=1$ and extended by Deny [**2**, pp. 144–145].

The conditions of Theorem 4 are clearly unnecessary since an ordinary set of uniqueness is of type $U^q$ for every $q$, $1\leqq q<\infty$. However the estimates cannot be improved.

In conclusion we mention one amusing problem for $r$-dimensional Euclidean space. Suppose $f \in L^p$ and vanishes outside a compact set. Then $\hat{f}$ is an entire function of exponential type. For $r=1$, the translates of $f$ span $L^p$ for all $p$, $1<p<\infty$, since $Z(f)$ is countable. However consideration of a few Bessel functions leads to the conclusion that for $r>1$ the theorem is certainly false unless $p \geqq 2r/(r+1)$. Is this a sufficient condition? Posing the problem otherwise, for what $q$ is the set of real zeros of an entire function of exponential type in $r$-variables necessarily of type $U^q$?

## Bibliography

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