ON SQUARE ROOTS OF NORMAL OPERATORS[1]

C. R. PUTNAM

1. All operators in this paper are bounded (linear, everywhere defined) transformations on a Hilbert space of elements x. An arbitrary operator A will be called a square root of a normal operator N if

$$A^2 = N. \tag{1}$$

It is clear that if N possesses the spectral resolution $N = \int z dK(z)$, then any operator of the form $A = \int z^{1/2} dK(z)$, where, for the value of $z^{1/2}$, the choice of the branch of the function may depend on z, is a solution of (1). Moreover, all such operators are even normal.

Of course, equation (1) may have other, nonnormal, solutions A. The object of this note is to point out a simple condition to be satisfied by a square root A guaranteeing that it be normal. This criterion will involve the (closed, convex) set $W = W_A$ consisting of the closure of the set of values (Ax, x) where $\|x\| = 1$. (Cf. also [2] wherein is discussed a connection between commutators and the set W.)

The following theorem will be proved:

(I) *Let N be a fixed normal operator and let A denote an arbitrary solution of* (1). *Suppose that there exists a line L in the complex plane passing through the origin and lying entirely on one side of* (*and possible lying all, or partly, in*) *the set W_A. Then A is necessarily normal.*

It is easy to see that the hypothesis of (I) concerning the line L is surely satisfied if W is a single point or a straight line segment. In this case, A is even the sum of multiples of a self-adjoint operator and the unit operator I. (In fact, there exists some angle θ and some complex number z such that the set W belonging to $e^{i\theta}A + zI$ is a point or a segment of the real axis, and hence $e^{i\theta}A + zI$ is self-adjoint.) In case the set W is actually two-dimensional, the assumption amounts to supposing that 0 is not in the interior of W, although it is allowed of course that 0 be on the boundary.

2. **Proof of** (I). Clearly, one can choose an angle θ for which the operator $B = e^{i\theta}A$ satisfies $B + B^* \geqq 0$. If $B = H + iJ$, where $H = (B + B^*)/2$ and $J = -i(B - B^*)/2$ denote the self-adjoint real and imaginary parts of B, then

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$$B^2 = e^{2i\theta}A^2 = (H^2 - J^2) + i(HJ + JH). \tag{2}$$

Since B^2 is normal and obviously commutes with B, it follows that B^2 also commutes with B^*; [1]. Consequently B^2 commutes with each of the operators H and J. A subtraction of the two relations obtained from (2) by a multiplication by H on the left and on the right, respectively, now implies $R+iS=0$, where $R=J^2H-HJ^2$ and $S=H^2J-JH^2$. On taking adjoints, one obtains $R-iS=0$. Therefore $S=0$, that is $H^2J=JH^2$; hence, since $H\geqq 0$, $HJ=JH$. Consequently B, hence A, is normal and the proof of (I) is now complete.

3. The following is a corollary of (I) and its proof:

(II) *Let N be a fixed self-adjoint operator and let A denote a solution of* (1) *for which either* (a) $\Re(Ax, x)\neq 0$ *or* (b) $\Im(Ax, x)\neq 0$ *holds for all x. Then either A or iA is self-adjoint according as* (a) *or* (b) *holds.*

It should be noted that the hypothesis of (II) implies that the line L of (I) can be chosen either as the imaginary axis or as the real axis according as (a) or (b) holds and that, moreover, no number (Ax, x), for $\|x\|=1$, actually lies on L (although, of course, such numbers may cluster at a point of L).

In order to prove (II), note that the angle θ occurring in the proof of (I) can now be chosen to be 0 or π in case (a) and $\pi/2$ or $3\pi/2$ in case (b). Furthermore, $(Hx, x)>0$ whenever $\|x\|=1$, so that 0 is not in the point spectrum of H. Since $e^{2i\theta}$ is real, it follows from the relation (2) that $HJ+JH=0$. This fact combined with the relation $HJ-JH=0$ implies $HJ=0$, hence $J=0$. Thus B $(=H)$ is self-adjoint and so $A=e^{-i\theta}B$. In view of the choice of θ, the proof of (II) is now complete.

References

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Purdue University