THE SPECTRA OF BOUNDED LINEAR OPERATORS ON THE SEQUENCE SPACES[1]

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These remarks are the result of an investigation into the connections among the spectra of the various operators defined on the sequence spaces l_p by the same infinite matrix.

We will have need of the following notation. Let $[l_p]$ denote the algebra of bounded linear operators mapping the sequence space l_p into itself. If A is an infinite matrix which defines an element of $[l_p]$ (we shall sometimes use the statement $A \in [l_p]$ to express this state of affairs), we shall denote that operator by A_p. We shall denote the transpose of the matrix A by A^t. Let $\|A_p\|$ denote the norm of the operator A_p on the sequence space l_p, and let $\sigma(A_p)$, $|\sigma(A_p)|$ and $\rho(A_p)$ denote its spectrum, spectral radius and resolvent set respectively. If T is the operator defined by the infinite matrix (t_{ij}) we shall denote by $\overline{T}$ the operator defined by the infinite matrix $(\bar{t}_{ij})$, where $\bar{z}$ denotes the complex conjugate of the complex number z. Similarly, if we have a vector $x=(\xi_1, \xi_2, \cdots)$, then $\bar{x}$ will denote the vector $(\bar{\xi}_1, \bar{\xi}_2, \cdots)$. The symbol $l_p(n)$ will denote the vector space of n-tuples of complex numbers, such that if $x=(\xi_1, \xi_2, \cdots, \xi_n) \in l_p(n)$, then $\|x\|_p = (\sum_{i=1}^n |\xi_i|^p)^{1/p}$. Given a number p, p' will denote the number $p/(p-1)$ if $1<p<\infty$, and will denote ∞ or 1 respectively, according as $p=1$ or $p=\infty$. The statement "q lies between p and p'" will mean $p \leqq q \leqq p'$ or $p' \leqq q \leqq p$ respectively, according as $1 \leqq p \leqq 2$ or $2 < p \leqq \infty$.

M. Riesz has proved that if A is a matrix transformation of $l_{1/\alpha}(n)$ into $l_{1/\beta}(m)$ and $M_{\alpha\beta}$ is its norm, then $M_{\alpha\beta}$ as a function of the point (α, β) has the property that $\log(M_{\alpha\beta})$ is a convex function in the triangle $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, $\alpha+\beta \leqq 1$, [2, pp. 466–471]. G. O. Thorin has shown that the theorem actually holds for the entire first quadrant of the $\alpha\beta$-plane, [4, pp. 5–6].

This result, with the use of a limiting process, implies the following inequality

$$\|A_{1/b}\|^{c-a} \leqq \|A_{1/a}\|^{c-b} \|A_{1/c}\|^{b-a} \tag{1}$$

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[1] The results of this paper are based in part on a University of California (at Los Angeles) thesis prepared under the direction of Professor Angus E. Taylor.

where $0 \leqq a \leqq b \leqq c$. (It is understood that if $a=0$, $1/a$ is replaced by ∞.) For the case where $1 \leqq p \leqq q \leqq r \leqq \infty$, this gives

$$(2) \qquad \|A_q\| \leqq \|A_p\|^{(r-q)p/(r-p)q}\|A_r\|^{(q-p)r/(r-p)q},$$

it being understood that if $r=\infty$, then $(r-q)/(r-p)=r/(r-p)=1$. (For the case where $p=1$, $q=2$, and $r=\infty$, this is simply a reformulation of a theorem due to J. Schur [**3**, p. 6].) Or, since it can easily be shown that $\|A_{p'}\|=\|(A^t)_p\|$, if we let $1 \leqq p \leqq \infty$ and let q lie between p and p' we can restate (2) as

$$(3) \qquad \|A_q\| \leqq \|A_p\|^{(p+q(1-p))/(2-p)q}\|(A^t)_p\|^{(q-p)/(2-p)q}.$$

We note that (2) implies that if $1 \leqq p \leqq q \leqq r \leqq \infty$ and A belongs to both $[l_p]$ and $[l_r]$, then $A \in [l_q]$ and that (3) implies that if $1 \leqq p \leqq \infty$, q lies between p and p', and both A and A^t belong to $[l_p]$, then $A \in [l_q]$.

Borrowing from the phraseology of M. Riesz, we state the following theorem.

THEOREM 1. *If $A \in [l_{1/\alpha}]$ for each α, $0 \leqq \alpha \leqq 1$, then $|\sigma(A_{1/\alpha})|$ as a function of the number α has the property that* $\log |\sigma(A_{1/\alpha})|$ *is a convex function for* $0 \leqq \alpha \leqq 1$.

PROOF. We know that $\lim_{n\to\infty} \|(A_{1/\alpha})^n\|^{1/n} = |\sigma(A_{1/\alpha})|$ provided $0 \leqq \alpha \leqq 1$, and (1) implies that

$$\|(A_{1/b})^n\|^{(c-a)/n} \leqq \|(A_{1/a})^n\|^{(c-b)/n}\|(A_{1/c})^n\|^{(b-a)/n},$$

for $0 \leqq a \leqq b \leqq c \leqq 1$. If we let n approach infinity, we obtain

$$(4) \qquad |\sigma(A_{1/b})|^{c-a} \leqq |\sigma(A_{1/a})|^{c-b}|\sigma(A_{1/c})|^{b-a},$$

which is the desired result.

When $1 \leqq p \leqq q \leqq r$ we can restate the inequality (4) as

$$(5) \qquad |\sigma(A_q)| \leqq |\sigma(A_p)|^{(r-q)p/(r-p)q}|\sigma(A_r)|^{(q-p)r/(r-p)q}$$

provided A belongs to both $[l_p]$ and $[l_r]$.

In the special case where $1 \leqq p \leqq \infty$ and q lies between p and p', we can use the well known fact that $\sigma(A_{p'})=\sigma((A^t)_p)$ together with the inequality (5) to derive the following inequality, provided both A and A^t belong to $[l_p]$.

$$(6) \qquad |\sigma(A_q)| \leqq |\sigma(A_p)|^{(p+q(1-p))/(2-p)q}|\sigma((A^t)_p)|^{(q-p)/(2-p)q}.$$

Using the inequality (5) we can immediately state the following theorem.

THEOREM 2. *Suppose that T belongs to both $[l_p]$ and $[l_r]$, that*

$1 \leqq p \leqq q \leqq r \leqq \infty$ *and that* $|\sigma(T_p)| \neq |\sigma(T_r)|$. *Then either*

$$|\sigma(T_p)| > |\sigma(T_q)| \quad \textit{or} \quad |\sigma(T_r)| > |\sigma(T_q)|$$

according as

$$|\sigma(T_p)| > |\sigma(T_r)| \quad \textit{or} \quad |\sigma(T_r)| > |\sigma(T_p)|,$$

respectively.

The inequality (6) together with fact that if q lies between p and p', then so does q', and the fact that $|\sigma(A_{q'})| = |\sigma((A^t)_q)|$ imply the following theorem.

THEOREM 3. *Let both T and T^t belong to $[l_p]$, $1 \leqq p \leqq \infty$, let q lie between p and p' and let $|\sigma(T_p)| \neq |\sigma((T^t)_p)|$. Then either*

$$|\sigma(T_p)| > |\sigma(T_q)| \quad \textit{or} \quad |\sigma((T^t)_p)| > |\sigma((T^t)_q)|$$

according as

$$|\sigma(T_p)| > |\sigma((T^t)_p)| \quad \textit{or} \quad |\sigma((T^t)_p)| > |\sigma(T_p)|,$$

respectively.

We shall now use the inequalities (2) and (3) to derive some set relationships among the spectra of the operators defined on the sequence spaces l_p by the same infinite matrix.

THEOREM 4. *Let T belong to both $[l_p]$ and $[l_r]$ and let $1 \leqq p \leqq q \leqq r \leqq \infty$. Then*

(a) $$\sigma(T_q) \subset \sigma(T_p) \cup \sigma(T_r),$$

and

(b) *if C is any component of $\sigma(T_q)$, then the set $C \cap (\sigma(T_p) \cap \sigma(T_r))$ is nonvoid.*

PROOF. (a) Assume that $\lambda \in \rho(T_p) \cap \rho(T_r)$. This implies that $(\lambda I - T)^{-1}$ belongs to both $[l_p]$ and $[l_r]$. From (2) we infer that $(\lambda I - T)^{-1} \in [l_q]$, whence $\lambda \in \rho(T_q)$. We have thus proved that

$$\rho(T_p) \cap \rho(T_r) \subset \rho(T_q),$$

whence, by complementation, we have

$$\sigma(T_p) \cup \sigma(T_r) \supset \sigma(T_q).$$

(b) Let us assume that $C \cap (\sigma(T_p) \cap \sigma(T_r))$ is void. It is clear that $C \cap \sigma(T_p)$ and $C \cap \sigma(T_r)$ are both closed, and by our assumption they have no point in common; moreover, it can be shown that they are both nonvoid [1, p. 288]. But

$$C = (C \cap \sigma(T_p)) \cup (C \cap \sigma(T_r))$$

since, by (a), $\sigma(T_q) \subset \sigma(T_p) \cup \sigma(T_r)$. We are therefore forced to conclude that C is not connected, which is in contradiction to our assumption that C is a component.

We note that (b) implies, among other things, that $\sigma(T_p)$ and $\sigma(T_r)$ always have points in common.

THEOREM 5. *Suppose that both T and T^t belong to $[l_p]$, $1 \leqq p \leqq \infty$, and q lies between p and p'. Then*

(a) $$\sigma(T_q) \subset \sigma(T_p) \cup \sigma((T^t)_p),$$

(b) $$\sigma(T_2) \subset \sigma(T_q) \cup \sigma((T^t)_q) \subset \sigma(T_p) \cup \sigma((T^t)_p),$$

and

(c) *if C is any component of $\sigma(T_q)$, then the set*

$$C \cap (\sigma(T_p) \cap \sigma((T^t)_p))$$

is nonvoid.

PROOF. The statements (a) and (c) are proved in the same manner as statements (a) and (b) of Theorem 4 except that the inequality (3) is used instead of the inequality (2). The first containment of the statement (b) results from letting $q=2$ in (a), and the second containment results from (a) combined with the application of (a) to T^t.

Combining the result (b) of Theorem 5 with the classical result that the spectrum of a bounded operator defined on l_2 by a hermitian symmetric matrix is real, we obtain the following theorem.

THEOREM 6. *If $T \in [l_p]$, $T = \overline{T}^t$ and $1 \leqq p \leqq \infty$ then*

$$\sigma(T_2) \subset \sigma(T_p).$$

PROOF. Since if $x \in l_p$, $x \rightarrow \bar{x}$ gives an isometric isomorphic mapping of l_p onto itself we see that if $T \in [l_p]$, then $\sigma(T_p) = \text{Conj}\ [\sigma(\overline{T}_p)]$. Thus, since by hypothesis, $(T^t)_p = \overline{T}_p$, we have $\sigma((T^t)_p) = \text{Conj}\ [\sigma(T_p)]$; whence result (b) of Theorem 5 implies that

$$\sigma(T_2) \subset \sigma(T_p) \cup \text{Conj}\ [\sigma(T_p)].$$

Knowing from a classical result that $T_2 = (\overline{T}^t)_2$ implies that $\sigma(T_2)$ is real, we are led to the desired conclusion.

Our final theorem was suggested by Professor Angus E. Taylor.

THEOREM 7. *If $1 \leqq p \leqq q$, T belongs to both l_p and l_q, and $\lambda \in \rho(T_q)$, then a necessary and sufficient condition that $\lambda \in \rho(T_p)$ is that*

$$(\lambda I - T)(l_q - l_p) \subset l_q - l_p.$$

PROOF. Suppose that $\lambda \in \rho(T_p)$. Then $\lambda I - T$ maps l_p onto l_p in a 1-1 manner; and since by hypothesis $\lambda \in \rho(T_q)$, $\lambda I - T$ maps l_q onto l_q in a 1-1 manner. Hence

$$(\lambda I - T)(l_q - l_p) = l_q - l_p.$$

Now suppose that $(\lambda I - T)(l_q - l_p) \subset l_q - l_p$. It follows that $(\lambda I - T)^{-1} l_p \supset l_p$, for if $x \in l_p$ and $(\lambda I - T)x = y$, then $y \in l_p$ and $(\lambda I - T)^{-1} y = x$, whence $x \in (\lambda I - T)^{-1} l_p$. If we now assume that $x \in ((\lambda I - T)^{-1}(l_p) - l_p)$, then $x = (\lambda I - T)^{-1} y$, where $y \in l_p$ and $x \notin l_p$. But $(\lambda I - T)x = y$, which implies that $y \in l_q - l_p$ by our assumption that $(\lambda I - T)(l_q - l_p) \subset l_q - l_p$. This is a contradiction and we are thus led to the conclusion that $(\lambda I - T)^{-1}(l_p) - l_p$ is void, and this with our earlier conclusion implies that $l_p = (\lambda I - T)^{-1} l_p$. Thus we see that $(\lambda I - T)^{-1}$ (and hence also $\lambda I - T$) sets up a 1-1 map of l_p onto l_p. From this it follows that $(\lambda I - T)^{-1}$ is bounded, for it is known that the inverse of a linear 1-1 map of a Banach space onto itself is bounded. The desired conclusion follows immediately.

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