ON POINCARÉ'S LAST GEOMETRICAL THEOREM

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1. Let $\boldsymbol{R}$ be an annulus with inner radius a and outer radius b; i.e. the set of all points (x, y) such that $a^2 \leqq r^2 = x^2 + y^2 \leqq b^2$. Let T be a homeomorphism of $\boldsymbol{R}$ onto itself, keeping the bounding circles fixed. If we introduce polar coordinates (r, θ), $(0 \leqq \theta < 2\pi)$ we may write $T(r, \theta) = (r', \theta')$ where θ' is determined mod 2π. For a particular point (r_0, θ_0) choose *some* determination θ_0^* of θ_0' and consider the function $\theta^* - \theta = \psi(\theta)$. This is clearly defined in some neighborhood of (r_0, θ_0) in virtue of the continuity of T. Furthermore $\psi(\theta)$ has a unique continuation along any path emanating from (r_0, θ_0). Since the change in value of $\psi(\theta)$ about any closed path clearly depends only on the homotopy class of this path, it suffices to evaluate it for the circle $r = a$. Since as the point p traverses the circle $r = a$ both θ and θ^* increase by exactly 2π we obtain the result that the function $\psi(\theta)$ is single valued. Choose the initial point (r_0, θ_0) to be on the circle $r = a$ and choose θ^* so that $\theta_0^* - \theta_0 \geqq 0$.

Suppose T has the property that $\psi(\theta)$ (for this choice of θ_0^*) is negative on $r = b$. Then a celebrated theorem of Poincaré [2] first proved by Birkhoff [1] states that if T is assumed to be measure preserving then T has a fix point. It was remarked by Wintner [3] that this theorem can be formulated in terms of absolute constants. That is, if T is no longer assumed to be measure preserving, let $d(T) = \min |TP - P|$, and $\alpha = \min_{P \in R} |TU| / |U|$ where $|U|$ is the area of U and U varies over all open subsets of the annulus. (In case T is a C^1 transformation α is the minimum of the Jacobian of T). The desired formulation of the theorem then asserts that there exists a functional relationship $d = d(\alpha; a, b)$ such that $d \to 0$ as $\alpha \to 1$. The purpose of this paper is to calculate the "best possible" value for the function $d(\alpha; a, b)$. The method that we use is essentially due to Birkhoff.

2. In order to study such homeomorphisms of the annulus it is convenient to introduce the "modified polar" coordinates $R = r^2 = x^2 + y^2$, $\theta = \theta$. This maps the annulus $a \leqq r \leqq b$ onto the strip $\boldsymbol{S}$, $A = a^2 \leqq R \leqq B = b^2$, $-\infty < \theta < \infty$ where θ is identified mod 2π. The mapping is measure preserving if one half the ordinary measure

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$dRd\theta$ is put on the strip. In fact $dRd\theta/2 = rdrd\theta = dxdy$. Furthermore a choice of the function ψ causes T to induce a unique transformation T^* on the strip S, The transformation T^* clearly has the same Jacobian as T at corresponding points and the existence of a fix point of T^* certainly implies that for T. We shall therefore restrict attention to the strip S and the transformation T^* and the function $\psi(T^*)$.

3. We now reformulate the problem in terms of the strip (omitting the asterisk). Let T be a homeomorphism of the strip $A \leqq R \leqq B$, $-\infty < \theta < \infty$ in the R, θ plane which keeps the lines $R = A$ and $R = B$ invariant. Furthermore, let $\theta' > \theta$ for $R = A$ and $\theta' < \theta$ for $R = B$ where $T(R, \theta) = (R', \theta')$. The transformation T induces a vector field $\mathcal{V}$ on the strip in a natural manner. Namely, to every point P we assign the vector $(\theta' - \theta, R' - R) = (u, v)$. The function $\delta(T) = \min |TP - P|$ is the distance of the image set of the vector field from the origin. For $R = A$ all the images of $\mathcal{V}$ line on the positive u-axis at a distance $\min_{R=A} |TP - P|$ from the origin. Similarly for $R = B$ the images of $\mathcal{V}$ lie on the negative u-axis a distance $\min_{R=B} |TP - P|$ from the origin. (Thus $\delta < \min_{R=A, R=B} |\theta' - \theta|$.) If $\delta \neq 0$ then we can define a unique function $i(p)$ which gives the angle of inclination of $\mathcal{V}(p)$ with respect to the u-axis. In fact, defining its value at one point p_0 arbitrarily, by the continuity of $\mathcal{V}$ and its nonvanishing we may continue it along any path. Since S is simply connected this yields a single valued function. The number $i = i(p^*) - i(p)$ for p on $R = A$ and p^* on $R = B$ is clearly independent of p, p^* and the initial choice of $i(p_0)$. It is obviously some odd multiple of π. Since the image of V is at a distance of at least δ from the origin, we may calculate i by adding a vector field $\mathcal{W}$ all of whose image vectors have length less than δ and computing the change in angle for $\mathcal{V} + \mathcal{W}$ and then the change from $\mathcal{V}$ to $\mathcal{V} + \mathcal{W}$. A convenient vector field $\mathcal{W}$ to choose is one corresponding to the vertical translation operator T_ϵ: $(R, \theta) \rightarrow (R + \epsilon, \theta)$, where $0 < \epsilon < \delta$. Since T_ϵ is area preserving, $T_\epsilon T$ has the same Jacobian as T. Now the transformation $T_\epsilon T$ takes the line $R = A$ into the line $R = A + \epsilon$. Applying $T_\epsilon T$ to the strip $A_0 = A \leqq R \leqq A + \epsilon$ we will obtain a nonoverlapping adjacent domain A_1 and so on under iteration. If we examine the restrictions B_n of the domain A_n $(n = 0, 1, 2, \cdots)$ to a fixed rectangle $0 \leqq \theta \leqq 2\pi$ of the strip S we find, by periodicity of $T_\epsilon T$ that $|B_n| \geqq 2\pi\epsilon(1 + \alpha + \alpha^2 + \cdots + \alpha^n)$ where $|B_n|$ is the area of B_n. (Here $2\pi\epsilon$ is the area of B_0.) Hence, if $\epsilon > (1 - \alpha)^{-1}(B - A)$ then, after a finite number of applications of $T_\epsilon T$, a point p on $R = A$ will be moved above $R = B$. If we join p to its image p_1 under $T_\epsilon T$ by a continuous arc and then iterate this arc, we will obtain a simple arc C joining a point p on $R = A$ to a point p' above $R = B$ with the property

that the vectors of $\mathcal{V}+\mathcal{W}$ on the arc C are all secant vectors. Now the vector of $\mathcal{V}+\mathcal{W}$ at the point p makes an angle $0<\theta<\pi/2$ with the u-axis (since $\epsilon<\delta$) and at the point p' makes an angle $\pi/2<\theta<\pi$. Since the angle from a vector of $\mathcal{V}$ to that of $\mathcal{V}'$ is always less than $\pi/2$, and since the change in angle of the secant vector to a simple arc is always less than 2π we obtain that $i=\pi$. Similarly, if we use $T_{-\epsilon}$ instead of T_ϵ we obtain that $i=-\pi$, a contradiction. Thus

$$\delta \leqq (1-\alpha)(B-A). \tag{1}$$

4. We now show that this is, in a sense, best possible. To do this we shall combine a purely radial displacement with a transformation rotating each concentric circle. In fact, let f_n be monotone increasing differentiable functions on the unit interval satisfying $f_n(0)=0$, $f_n(1)=1$, $f'(x)\geqq C$ and which converges (uniformly in any compact subset of $[0, 1]$) to the function $g(x)=Cx$. Let x_n be a point where the function $x-f_n(x)$ assumes a maximum. Choose ξ_n so that $|f(x)-f(x_n)|\to 0$ for all x such that $|x-x_n|\leqq\xi_n$. Let ϕ_n be a smooth function on $(0, 1)$ constant outside the set $|x-x_n|\leqq\xi_n$ and such that $\phi_n(0)=\pi$, $\phi_n(1)=-\pi$. Now the transformation of the strip $0\leqq R\leqq 1$

$$T_n^1:(R, \theta)\to(R, \theta+\phi_n(R))$$

is clearly area preserving. The transformation

$$T_n^2:(R, \theta)\to(f_n(R), \theta)$$

has the property that $\alpha(T_n^2)$ is always greater than C. Consider the transformation $T_n=T_n^2T_n^1$. A point not in the strip $|R-x_n|<\xi_n$ is moved a distance at least π. A point in the strip $|R-x_n|<\xi_n$ is moved a distance of at least δ_n where $\delta_n\to 1-C$. Since $\alpha(T^n)=\alpha(T_n^2)$ we see that (1) is best possible.

The results obtained above are for the strip. They can be transferred to the circle by the mapping given in paragraph 2 with suitable modification of formula (1).

References

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