

# ON POINCARÉ'S LAST GEOMETRICAL THEOREM

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1. Let  $R$  be an annulus with inner radius  $a$  and outer radius  $b$ ; i.e. the set of all points  $(x, y)$  such that  $a^2 \leq r^2 = x^2 + y^2 \leq b^2$ . Let  $T$  be a homeomorphism of  $R$  onto itself, keeping the bounding circles fixed. If we introduce polar coordinates  $(r, \theta)$ ,  $(0 \leq \theta < 2\pi)$  we may write  $T(r, \theta) = (r', \theta')$  where  $\theta'$  is determined mod  $2\pi$ . For a particular point  $(r_0, \theta_0)$  choose *some* determination  $\theta_0^*$  of  $\theta_0'$  and consider the function  $\theta^* - \theta = \psi(\theta)$ . This is clearly defined in some neighborhood of  $(r_0, \theta_0)$  in virtue of the continuity of  $T$ . Furthermore  $\psi(\theta)$  has a unique continuation along any path emanating from  $(r_0, \theta_0)$ . Since the change in value of  $\psi(\theta)$  about any closed path clearly depends only on the homotopy class of this path, it suffices to evaluate it for the circle  $r = a$ . Since as the point  $p$  traverses the circle  $r = a$  both  $\theta$  and  $\theta^*$  increase by exactly  $2\pi$  we obtain the result that the function  $\psi(\theta)$  is single valued. Choose the initial point  $(r_0, \theta_0)$  to be on the circle  $r = a$  and choose  $\theta^*$  so that  $\theta_0^* - \theta_0 \geq 0$ .

Suppose  $T$  has the property that  $\psi(\theta)$  (for this choice of  $\theta_0^*$ ) is negative on  $r = b$ . Then a celebrated theorem of Poincaré [2] first proved by Birkhoff [1] states that if  $T$  is assumed to be measure preserving then  $T$  has a fix point. It was remarked by Wintner [3] that this theorem can be formulated in terms of absolute constants. That is, if  $T$  is no longer assumed to be measure preserving, let  $d(T) = \min |TP - P|$ , and  $\alpha = \min_{P \in R} |TU|/|U|$  where  $|U|$  is the area of  $U$  and  $U$  varies over all open subsets of the annulus. (In case  $T$  is a  $C^1$  transformation  $\alpha$  is the minimum of the Jacobian of  $T$ ). The desired formulation of the theorem then asserts that there exists a functional relationship  $d = d(\alpha; a, b)$  such that  $d \rightarrow 0$  as  $\alpha \rightarrow 1$ . The purpose of this paper is to calculate the "best possible" value for the function  $d(\alpha; a, b)$ . The method that we use is essentially due to Birkhoff.

2. In order to study such homeomorphisms of the annulus it is convenient to introduce the "modified polar" coordinates  $R = r^2 = x^2 + y^2$ ,  $\theta = \theta$ . This maps the annulus  $a \leq r \leq b$  onto the strip  $S$ ,  $A = a^2 \leq R \leq B = b^2$ ,  $-\infty < \theta < \infty$  where  $\theta$  is identified mod  $2\pi$ . The mapping is measure preserving if one half the ordinary measure

Received by the editors September 25, 1956 and, in revised form, November 16, 1956.

$dRd\theta$  is put on the strip. In fact  $dRd\theta/2 = r dr d\theta = dx dy$ . Furthermore a choice of the function  $\psi$  causes  $T$  to induce a unique transformation  $T^*$  on the strip  $S$ . The transformation  $T^*$  clearly has the same Jacobian as  $T$  at corresponding points and the existence of a fix point of  $T^*$  certainly implies that for  $T$ . We shall therefore restrict attention to the strip  $S$  and the transformation  $T^*$  and the function  $\psi(T^*)$ .

3. We now reformulate the problem in terms of the strip (omitting the asterisk). Let  $T$  be a homeomorphism of the strip  $A \leq R \leq B$ ,  $-\infty < \theta < \infty$  in the  $R, \theta$  plane which keeps the lines  $R=A$  and  $R=B$  invariant. Furthermore, let  $\theta' > \theta$  for  $R=A$  and  $\theta' < \theta$  for  $R=B$  where  $T(R, \theta) = (R', \theta')$ . The transformation  $T$  induces a vector field  $\mathcal{U}$  on the strip in a natural manner. Namely, to every point  $P$  we assign the vector  $(\theta' - \theta, R' - R) = (u, v)$ . The function  $\delta(T) = \min |TP - P|$  is the distance of the image set of the vector field from the origin. For  $R=A$  all the images of  $\mathcal{U}$  line on the positive  $u$ -axis at a distance  $\min_{R=A} |TP - P|$  from the origin. Similarly for  $R=B$  the images of  $\mathcal{U}$  lie on the negative  $u$ -axis a distance  $\min_{R=B} |TP - P|$  from the origin. (Thus  $\delta < \min_{R=A, R=B} |\theta' - \theta|$ .) If  $\delta \neq 0$  then we can define a unique function  $i(p)$  which gives the angle of inclination of  $\mathcal{U}(p)$  with respect to the  $u$ -axis. In fact, defining its value at one point  $p_0$  arbitrarily, by the continuity of  $\mathcal{U}$  and its nonvanishing we may continue it along any path. Since  $S$  is simply connected this yields a single valued function. The number  $i = i(p^*) - i(p)$  for  $p$  on  $R=A$  and  $p^*$  on  $R=B$  is clearly independent of  $p, p^*$  and the initial choice of  $i(p_0)$ . It is obviously some odd multiple of  $\pi$ . Since the image of  $V$  is at a distance of at least  $\delta$  from the origin, we may calculate  $i$  by adding a vector field  $\mathcal{W}$  all of whose image vectors have length less than  $\delta$  and computing the change in angle for  $\mathcal{U} + \mathcal{W}$  and then the change from  $\mathcal{U}$  to  $\mathcal{U} + \mathcal{W}$ . A convenient vector field  $\mathcal{W}$  to choose is one corresponding to the vertical translation operator  $T_\epsilon: (R, \theta) \rightarrow (R + \epsilon, \theta)$ , where  $0 < \epsilon < \delta$ . Since  $T_\epsilon$  is area preserving,  $T_\epsilon T$  has the same Jacobian as  $T$ . Now the transformation  $T_\epsilon T$  takes the line  $R=A$  into the line  $R=A + \epsilon$ . Applying  $T_\epsilon T$  to the strip  $A_0 = A \leq R \leq A + \epsilon$  we will obtain a nonoverlapping adjacent domain  $A_1$  and so on under iteration. If we examine the restrictions  $B_n$  of the domain  $A_n$  ( $n=0, 1, 2, \dots$ ) to a fixed rectangle  $0 \leq \theta \leq 2\pi$  of the strip  $S$  we find, by periodicity of  $T_\epsilon T$  that  $|B_n| \geq 2\pi\epsilon(1 + \alpha + \alpha^2 + \dots + \alpha^n)$  where  $|B_n|$  is the area of  $B_n$ . (Here  $2\pi\epsilon$  is the area of  $B_0$ .) Hence, if  $\epsilon > (1 - \alpha)^{-1}(B - A)$  then, after a finite number of applications of  $T_\epsilon T$ , a point  $p$  on  $R=A$  will be moved above  $R=B$ . If we join  $p$  to its image  $p_1$  under  $T_\epsilon T$  by a continuous arc and then iterate this arc, we will obtain a simple arc  $C$  joining a point  $p$  on  $R=A$  to a point  $p'$  above  $R=B$  with the property

that the vectors of  $\mathfrak{U} + \mathfrak{W}$  on the arc  $C$  are all secant vectors. Now the vector of  $\mathfrak{U} + \mathfrak{W}$  at the point  $p$  makes an angle  $0 < \theta < \pi/2$  with the  $u$ -axis (since  $\epsilon < \delta$ ) and at the point  $p'$  makes an angle  $\pi/2 < \theta < \pi$ . Since the angle from a vector of  $\mathfrak{U}$  to that of  $\mathfrak{U}'$  is always less than  $\pi/2$ , and since the change in angle of the secant vector to a simple arc is always less than  $2\pi$  we obtain that  $i = \pi$ . Similarly, if we use  $T_{-\epsilon}$  instead of  $T_{\epsilon}$  we obtain that  $i = -\pi$ , a contradiction. Thus

$$(1) \quad \delta \leq (1 - \alpha)(B - A).$$

4. We now show that this is, in a sense, best possible. To do this we shall combine a purely radial displacement with a transformation rotating each concentric circle. In fact, let  $f_n$  be monotone increasing differentiable functions on the unit interval satisfying  $f_n(0) = 0$ ,  $f_n(1) = 1$ ,  $f'_n(x) \geq C$  and which converges (uniformly in any compact subset of  $[0, 1]$ ) to the function  $g(x) = Cx$ . Let  $x_n$  be a point where the function  $x - f_n(x)$  assumes a maximum. Choose  $\xi_n$  so that  $|f(x) - f(x_n)| \rightarrow 0$  for all  $x$  such that  $|x - x_n| \leq \xi_n$ . Let  $\phi_n$  be a smooth function on  $(0, 1)$  constant outside the set  $|x - x_n| \leq \xi_n$  and such that  $\phi_n(0) = \pi$ ,  $\phi_n(1) = -\pi$ . Now the transformation of the strip  $0 \leq R \leq 1$

$$T_n^1: (R, \theta) \rightarrow (R, \theta + \phi_n(R))$$

is clearly area preserving. The transformation

$$T_n^2: (R, \theta) \rightarrow (f_n(R), \theta)$$

has the property that  $\alpha(T_n^2)$  is always greater than  $C$ . Consider the transformation  $T_n = T_n^2 T_n^1$ . A point not in the strip  $|R - x_n| < \xi_n$  is moved a distance at least  $\pi$ . A point in the strip  $|R - x_n| < \xi_n$  is moved a distance of at least  $\delta_n$  where  $\delta_n \rightarrow 1 - C$ . Since  $\alpha(T_n) = \alpha(T_n^2)$  we see that (1) is best possible.

The results obtained above are for the strip. They can be transferred to the circle by the mapping given in paragraph 2 with suitable modification of formula (1).

#### REFERENCES

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