ON A CLASS OF LATTICE-ORDERED RINGS

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1. Introduction. It is the purpose of this paper to study a class of rings called F-rings. An *F-ring* R is a σ-complete vector lattice (Birkhoff [2, p. 238]) which is, in addition, a commutative algebra with a unit 1, satisfying the conditions

$$1 \geqq 0; \qquad x \geqq 0,\ y \geqq 0 \Rightarrow xy \geqq 0; \qquad x \wedge 1 = 0 \Rightarrow x = 0.$$

Here $\wedge$ denotes, as usual, the lattice operation greatest lower bound, and x, y are elements of R. A *bounded F-ring* is an F-ring $\overline{R}$ such that each $x \in \overline{R}$ satisfies

$$x \vee 0 + (-x) \vee 0 \leqq \lambda \cdot 1 \tag{1.1}$$

for some real number λ, the symbol $\vee$ denoting the lattice least upper bound.

Any ring R is regular [10] if for each $x \in R$ there is an $x^0 \in R$ such that $xx^0x = x$. It is evident that every regular F-ring R contains a maximal bounded sub-F-ring $\overline{R}$, the F-ring of all $x \in R$ satisfying equation (1.1). The relationship between a regular F-ring and its maximal bounded sub-F-ring is analogous to that between the ring of all continuous functions on a completely regular space X and the ring of all bounded continuous functions on X. For example, it is shown in Theorem 3 that there is a one-to-one correspondence between the maximal ideals of R and those of $\overline{R}$. (For the theory of rings of continuous functions, see [5] and [6].)

A maximal ideal M of a ring R is *real* [6] if the quotient ring $R - M$ is ring-isomorphic to the real field. An ideal S of an F-ring R is *closed* if $a_n \in S$, $n \geqq 1$ and $\bigvee_{n=1}^{\infty} a_n \in R$ imply $\bigvee_{n=1}^{\infty} a_n \in S$. It is proved in Theorems 5 and 6 that the closed maximal ideals of a regular F-ring are real and that there is a one-to-one correspondence between the closed maximal ideals of a regular F-ring R and the closed maximal ideals of $\overline{R}$, the maximal bounded sub-F-ring of R.

It is a direct corollary of some results of Nakano [9, pp. 39, 212] that a bounded F-ring is ring- and lattice-isomorphic to the ring of all continuous functions on a compact Hausdorff space. Therefore every bounded F-ring is a semisimple real Banach algebra. "Real" is used here in the classical sense, that is, a partially ordered ring R is

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real provided every element $a \geqq \epsilon$ for some real $\epsilon > 0$ has an inverse in R. (The notation ϵ is used in place of $\epsilon \cdot 1$.) An F-ring is called an M-ring if the intersection of all its real closed maximal ideals is the 0-element. A nontrivial example of a regular M-ring is the ring of all continuous functions on a P-space [**4**]. If Φ is an abstract set and $\mathfrak{K}$ is a σ-algebra of subsets of Φ, then a real function $f(\xi)$ defined on Φ is said to be *measurable* or $(\Phi, \mathfrak{K})$-*measurable* if for each real λ the set $\{\xi \mid f(\xi) \leqq \lambda\}$ belongs to $\mathfrak{K}$.

The main result of this paper (Theorem 7) is that any real M-ring B is ring- and lattice-isomorphic to $B(\Omega, \mathfrak{A})$, an F-ring of $(\Omega, \mathfrak{A})$-measurable functions. Here Ω designates the set of real closed maximal ideals of B, and $\mathfrak{A}$ is a σ-algebra of subsets of Ω isomorphic to the Boolean algebra I of idempotents of B. In addition (corollary to Theorem 8), if B is a regular, then $B(\Omega, \mathfrak{A})$ is *the* F-ring of all $(\Omega, \mathfrak{A})$-measurable functions.

2. **Notation and properties of F-rings.** In what follows, R always denotes a regular F-ring, $\overline{R}$ denotes its maximal bounded sub-F-ring, and I denotes the set of idempotents of R. Latin letters denote elements of rings, and Greek letters denote real numbers. Additional definitions are

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0, \qquad |x| = x^+ + x^-,$$

$$\bar{e}_x = \bigvee_{n=1}^{\infty} n|x| \wedge 1, \quad \text{and} \quad e_x = 1 - \bar{e}_x.$$

The following properties of F-rings (Nakano [**9**, Chapters I and IV]) are used throughout:

(N 1) For every $b \in R$, $a = \bigvee_{\lambda \in \Lambda} a_\lambda$ implies

$$a \wedge b = \bigvee_{\lambda \in \Lambda} (a_\lambda \wedge b)$$

and the dual statement, that is, the statement with $\vee$'s and $\wedge$'s replaced respectively by $\wedge$'s and $\vee$'s.

(N 2) If $a = \bigwedge_{\lambda \in \Lambda} a_\lambda$, then $a + b = \bigwedge_{\lambda \in \Lambda} (a_\lambda + b)$ and if $a = \bigvee_{\lambda \in \Lambda} a_\lambda$, then $a + b = \bigvee_{\lambda \in \Lambda} (a_\lambda + b)$ for all $b \in R$.

(N 3) If $a = \bigwedge_{\lambda \in \Lambda} a_\lambda$, then $ab = \bigwedge_{\lambda \in \Lambda} a_\lambda b$, and if $a = \bigvee_{\lambda \in \Lambda} a_\lambda$, then $ab = \bigvee_{\lambda \in \Lambda} a_\lambda b$, for all $b \geqq 0$.

(N 4) For all $a \in R$, $a^2 \geqq 0$.

(N 5) If $a \geqq 0$ and $b \geqq 0$, then $a \wedge b = 0$ is equivalent to $ab = 0$.

(N 6) R is archimedian, that is, for every non-negative element $a \in R$, $\bigwedge_{n=1}^{\infty} (1/n)a = 0$.

From (N 5), an element $e \in R$ is an idempotent if and only if $e \wedge (1 - e) = 0$. By the methods of [**8**], I forms a Boolean algebra with respect to the lattice operations of R. Since for $a, b \in I$, $ab \leqq a \wedge b$, and by (N 3), $a \wedge b = (a \wedge b)^2 = (a \wedge b) \wedge ab$, it follows that $ab = a \wedge b$. In addition, a further application of (N 3) shows that I is σ-complete.

From some results in [8, p. 530], it follows that

(N 7) $\bar{e}_x$ is idempotent.

In the sequel, (P) will denote the ideal of R generated by the subset P of R, and (x) will denote the ideal generated by $x \in R$.

3. **Regular F-rings.** This section begins with a proof of a general result concerning regular rings.

THEOREM 1. *Let A be a commutative ring with a unit* 1. *A is regular if and only if it has property*

(α) *For each $x \in A$ there exists an element $a_x \in A$ such that, first, $a_x^2 = a_x$, second, $xa_x = 0$, third, $x + a_x$ has an inverse.*

If x^0 is the element postulated in the definition of regularity, then $a_x = 1 - xx^0$.

PROOF. If A has property (α), then there exists $y \in A$ such that $y(x + a_x) = 1$. Therefore $x = x \cdot 1 = xy(x + a_x) = xyx$; that is, A is regular.

If A is regular, then there is an element $x^0 \in A$ such that $xx^0x = x$. It then follows, first, that xx^0 is an idempotent of A, second, that $x(1 - xx^0) = 0$, third, that $1 - xx^0$ is idempotent, and, fourth, that

$$[x + (1 - xx^0)][x(x^0)^2 - xx^0 + 1] = 1.$$

Thus with $a_x = 1 - xx^0$, A has property (α).

THEOREM 2. *If $x^0 \in R$ has the property $xx^0x = x$, then $xx^0 = \bar{e}_x$ and $1 - xx^0 = e_x$.*

PROOF. This consists in showing that $x\bar{e}_x = x$ and that $\bar{e}_x \in (x)$. From $x\bar{e}_x = x$, it follows that $(\bar{e}_x x)x^0 = xx^0$, and from $\bar{e}_x \in (x)$ it follows that $\bar{e}_x = xy$ for some $y \in A$, from which we deduce $\bar{e}_x xx^0 = xyxx^0 = xy$. Thus $\bar{e}_x = xx^0$.

To show that $\bar{e}_x x = x$, note that

$$x^+ \wedge \bar{e}_x = \bigvee_{n=1}^{\infty} x^+ \wedge n|x| \wedge 1 = x^+ \wedge 1 \tag{2.1}$$

and that, by the same reasoning,

$$x^- \wedge \bar{e}_x = x^- \wedge 1. \tag{2.2}$$

Since by (N 7) $\bar{e}_x$ is idempotent, equations (2.1) and (2.2) imply

$$0 = x^+ \wedge \bar{e}_x \wedge (1 - \bar{e}_x) = x^+ \wedge 1 \wedge (1 - \bar{e}_x) = x^+ \wedge (1 - \bar{e}_x) \tag{2.3}$$

and, by a similar line of reasoning, it follows that

$$(2.4) \qquad 0 = x^- \wedge (1 - \bar{e}_x).$$

From (N 5), (2.3), and (2.4), it follows that $x^+ \cdot \bar{e}_x = x^+$ and $x^- \cdot \bar{e}_x = x^-$, and hence $x\bar{e}_x = x$.

Finally we consider the proposition $\bar{e}_x \in (x)$. Since R is regular, the principal ideals (x), $(|x|)$, (xx^0) are all equal. By (N 4), $xx^0 \geqq 0$ and $1 - xx^0 \geqq 0$, and by (N 5), $|x| \cdot (1 - xx^0) = 0$. Hence (x) is closed under countable sup's (see (N 3)) and (x) is an l-ideal in the sense of Birkhoff [2, p. 222], that is, if $|y| \leqq |x|$, then $y \in (x)$. Since $n|x| \in (x)$, and since (x) is an l-ideal, it follows that $n|x| \wedge 1 \in (x)$ and $\bigvee_{n=1}^{\infty} n|x| \wedge 1 = \bar{e}_x \in (x)$.

COROLLARY. *R is real.*

PROOF. If $x \geqq \epsilon > 0$, then $n|x| \wedge 1 \geqq n\epsilon \wedge 1$ and hence $\bar{e}_x = 1$. Since $xx^0 = \bar{e}_x$, it follows (Theorem 1) that $e_x = 1 - xx^0 = 0$ and that x^{-1} exists.

COROLLARY. *If B is a partially ordered subring of R containing $\overline{R}$, then B is real.*

PROOF. If $x \in B$ and $x \geqq \epsilon > 0$, then $(x + e_x)^{-1} = x^{-1} \in R$ and $0 < x^{-1} \leqq 1/\epsilon$. Therefore $x^{-1} \in \overline{R} \subseteq B$.

4. **Maximal ideals of regular F-rings.** This section is devoted to a discussion of the relationship between the maximal (ring) ideals of R and those of $\overline{R}$.

THEOREM 3. *There is a one-to-one correspondence $(M \rightarrow \phi(M))$ between the maximal ideals of R and those of $\overline{R}$.*

PROOF. *Definition of ϕ*: Since $\overline{R}$ is ring- and lattice-isomorphic to the ring of continuous functions on a compact Hausdorff space, a result of Gillman and Henriksen [4, Theorem 3.3] shows that each prime ideal $\overline{P}$ of $\overline{R}$ is contained in a unique maximal ideal of R. If M is a maximal ideal of R, then $\overline{R} \cap M$ is a prime ideal of $\overline{R}$ and is contained in a unique maximal ideal $\phi(M)$ of $\overline{R}$. The mapping ϕ is then a single valued mapping of the maximal ideals of R into the maximal ideals of $\overline{R}$.

To show ϕ is a mapping *onto* the maximal ideals of $\overline{R}$, suppose $\overline{M}$ is a maximal ideal of $\overline{R}$. Then $\overline{M} \cap I$ is a prime ideal of the Boolean algebra I. Since R is regular and commutative, a result of Morrison [7] states that $(\overline{M} \cap I)$ is a maximal ideal of R. The prime ideal $(\overline{M} \cap I) \cap \overline{R}$ of $\overline{R}$ contains $\overline{M} \cap I$, so it must be contained in $\overline{M} = \phi[(\overline{M} \cap I)]$.

To establish that ϕ is biunique, it is first necessary to show that if

$\overline{P}$ is a prime ideal of $\overline{R}$, then either $(\overline{P})$ is equal to R or it is a maximal ideal. Indeed, suppose $(\overline{P})$ is different from R. Let a be an arbitrary element of R not in $(\overline{P})$. The ideal $(a, (\overline{P}))$, generated by a and $(\overline{P})$, contains $a+e_a$. For if $a \notin (\overline{P})$, then $\bar{e}_a \notin (\overline{P})$ because $a \cdot \bar{e}_a = a$. Since $\bar{e}_a$ does not belong to $\overline{P}$ either, it follows that $1-\bar{e}_a = e_a \in \overline{P}$. The regularity of R implies that $a+e_a$ possesses an inverse, so $(a, (\overline{P}))$ equals R and $(\overline{P})$ is a maximal ideal. If M_1 and M_2 are maximal ideals of R where $M_1 \cap \overline{R}$ and $M_2 \cap \overline{R}$ are both subsets of the same maximal ideal $\overline{M}$ of $\overline{R}$, then $\overline{M} \cap I = M_1 \cap \overline{R} \cap I = M_2 \cap \overline{R} \cap I$ is a prime ideal of I. Therefore from [**7**] it follows that $M_1 = M_2 = (\overline{M} \cap I)$.

As in the case of the ring of all continuous functions on a completely regular space (see [**6**] for example), there is no guarantee that all maximal ideals of R are real. Real maximal ideals are characterized by the following theorem.

THEOREM 4. *A necessary and sufficient condition for a maximal ideal M of R to be real is that $M \cap \overline{R}$ be a maximal ideal of $\overline{R}$.*

PROOF. If M is a real maximal ideal of R, then

$$R - M \supseteq [\overline{R} + M] - M \cong \overline{R} - M \cap \overline{R} \tag{4.1}$$

by the second homomorphism theorem for rings. The left-hand member is isomorphic to the real field and the right-hand member contains a field isomorphic to the real field. Therefore $\overline{R} - M \cap \overline{R}$ is isomorphic to the real field; hence $M \cap \overline{R}$ is a maximal ideal.

Let M be a maximal ideal of R. If $M \cap \overline{R}$ is a maximal ideal of $\overline{R}$, then formula (4.1) implies that $[\overline{R}+M]-M$ is isomorphic to the real field.

In order to finish the proof, it suffices to show that $\overline{R}+M=R$. The following inequality can be proved for each pair of real numbers $\lambda<\mu$ and each $x \in R$, using the fact that $\{e_{(x-\lambda)^+}\}$ is a spectral decomposition of 1 relative to x [**2**, p. 251] and that $xy=0$ implies $e_x y = y$. If $e_x(\lambda, \mu)$ stands for $e_{(x-\mu)^+} - e_{(x-\lambda)^+}$, then the inequality can be expressed as follows

$$\lambda e_x(\lambda, \mu) \leqq x e_x(\lambda, \mu) \leqq \mu e_x(\lambda, \mu). \tag{4.2}$$

Therefore $xe_x(\lambda, \mu) \in \overline{R}$.

Let x be an element of R not in either M or $\overline{R}$. Suppose $xe_x(\lambda, \mu) \in M \cap \overline{R}$ for all λ, μ $(\lambda<\mu)$. Then $e_x(\lambda, \mu) \in M \cap \overline{R}$ for all such λ, μ and in addition

$$a = \sum_{N=1}^{\infty} \frac{1}{N^2} e_x(N-1, N) + \sum_{N=1}^{\infty} \frac{1}{N^2} e_x(-N, -N+1) \tag{4.3}$$

belongs to $M \cap \bar{R}$ because the maximal ideals of the real Banach algebra $\bar{R}$ are norm-closed.

Since $\{e_{(x-\lambda)^+}\}$ is a spectral decomposition,

$$e_x(\lambda, \mu) \wedge e_x(\sigma, \tau) = e_x(\lambda, \mu) \cdot e_x(\sigma, \tau) = 0$$

if the closed intervals $[\lambda, \mu]$ and $[\sigma, \tau]$ have no more than one point in common. Therefore equation (4.3) can be replaced [9, Theorem 5.15] by

$$a = \bigvee_{N=1}^{\infty} \frac{e_x(N-1, N)}{N^2} \bigvee_{M=1}^{\infty} \frac{e_x(-M, -M+1)}{M^2}. \tag{4.4}$$

Now

$$\begin{aligned}
\bar{e}_a &= \bigvee_{n=1}^{\infty} na \wedge 1 \\
&= \bigvee_{n=1}^{\infty} \left[\bigvee_{N=1}^{\infty} \frac{n}{N^2} e_x(N-1, N) \vee \bigvee_{M=1}^{\infty} \frac{n}{M^2} e_x(-M, -M+1) \right] \wedge 1 \\
&= \bigvee_{n,N,M} \left[\frac{n}{N^2M^2} e_x(N-1, N) \vee e_x(-M, -M+1) \right] \wedge 1 \\
&= \bigvee_{N,M} \left\{ \bigvee_{n} \left[\frac{n}{N^2M^2} e_x(N-1, N) \vee e_x(-M, -M+1) \right] \wedge 1 \right\}
\end{aligned}$$

and because $e = e^2$ implies $ne \wedge 1 = e$ for all $n \geqq 1$, it follows that

$$\begin{aligned}
\bar{e}_a &= \bigvee_{N,M} [e_x(N-1, N) \vee e_x(-M, -M+1)] \\
&= 1.
\end{aligned}$$

Therefore $e_a = 0$ and a^{-1} belongs to R. Since this is impossible because $a \in M$, the supposition that $e_x(\lambda, \mu) \in M$ for all pairs (λ, μ) is incorrect.

Let λ and μ be numbers such that $e_x(\lambda, \mu) \notin M$. Then $1 - e_x(\lambda, \mu) \in M \cap \bar{R}$. For any $x \in R$, the element $xe_x(\lambda, \mu)$ belongs to $\bar{R}$, and $x - xe_x(\lambda, \mu)$ belongs to M. Thus $R = \bar{R} + M$, which concludes the proof.

Closed maximal ideals figure importantly in what follows. We therefore conclude this section with a few facts about them.

THEOREM 5. *The closed maximal ideals of R are real.*

PROOF. In order to show that a closed maximal ideal M is real, it suffices to show that $R - M$ is the real field. In the course of the proof of Theorem 2, it was shown that (x), and hence any maximal ideal M of R, is an l-ideal. Thus the quotient space $R - M$ is an l-group

[2, pp. 214, 222]. If $a(M)$ stands for the image of $a \in R$ under the natural homomorphism of R onto $R-M$, then $a^+(M) \cdot a^-(M) = 0$. Since $R-M$ is a field, either $a^+(M) = 0$ or $a^-(M) = 0$; hence $R-M$ is simply ordered. We know that $R-M$ is an ordered field because the following statement is a trivial consequence of the definition of order in $R-M$: $a(M) \geqq 0$ and $b(M) \geqq 0$ imply $a(M) \cdot b(M) = 0$.

That $R-M$ is a σ-complete vector lattice follows from the hypothesis that M is a closed maximal ideal. Under these circumstances, $R-M$ is archimedian (N 6) and hence it is isomorphic to the real field [2, Ex. 2, p. 229].

THEOREM 6. *There is a one-to-one correspondence between the closed maximal ideals of R and those of $\overline{R}$.*

PROOF. If M is a closed ideal of R, then $\phi(M) = M \cap \overline{R}$ is a maximal ideal of $\overline{R}$ (Theorems 4 and 5). The ideal $M \cap \overline{R}$ is also easily seen to be closed.

If, on the other hand, $\overline{M}$ is a closed maximal ideal of $\overline{R}$, then $\overline{M} \cap I$ is a σ-prime ideal of the Boolean algebra I, that is, $\overline{M} \cap I$ is a prime ideal of I satisfying the added property: if $a_n \in \overline{M} \cap I$ for all integers $n \geqq 1$ and $a = \bigvee_{n=1}^{\infty} a_n \in I$, then $a \in \overline{M} \cap I$. The ideal $(\overline{M} \cap I)$ of R is, by Morrison's Theorem [7], a maximal ideal. In addition, $(\overline{M} \cap I) \cap \overline{R} = \overline{M}$.

To finish the proof it suffices to show that $(\overline{M} \cap I)$ is a closed ideal of R. Suppose $x_n \in (\overline{M} \cap I)$ for $n \geqq 1$ and suppose $x = \bigvee_{n=1}^{\infty} x_n$ belongs to R. Then $\bar{e}_{x_n} \in \overline{M} \cap I$ for $n \geqq 1$ and because $\bar{e}_{x_n^+} \leqq \bar{e}_{x_n}$ (easily verifiable),

$$\bigvee_{m=1}^{\infty} \bar{e}_{x_m^+} = \bigvee_{m=1}^{\infty} \bigvee_{n=1}^{\infty} n x_m^+ \wedge 1 = \bigvee_{n=1}^{\infty} n\left(\bigvee_{m=1}^{\infty} x_m^+\right) \wedge 1.$$

Therefore $\bar{e}_{x^+} \in \overline{M} \cap I$, and also $x^+ \in (\overline{M} \cap I)$. That x_n^- for $n \geqq 1$ and $x^- = \bigwedge_{n=1}^{\infty} x_n^-$ all belong to $(\overline{M} \cap I)$ follows because $(\overline{M} \cap I)$ is an l-ideal. Thus $x^+ - x^- = x \in (\overline{M} \cap I)$; hence $(\overline{M} \cap I)$ is closed.

5. **Representation theorems for certain F-rings.** It is clear that the ring of all $(\Phi, \mathfrak{R})$-measurable functions is a regular M-ring. Indeed, each point $\xi \in \Phi$ corresponds to the closed maximal ideal of all functions vanishing at ξ, so the only function common to all closed maximal ideals is the zero function. The ring of all $(\Phi, \mathfrak{R})$-measurable functions contains every F-ring of $(\Phi, \mathfrak{R})$-measurable functions. In this section it is shown that, conversely, every regular M-ring is ring- and lattice-isomorphic to the M-ring of all $(\Phi, \mathfrak{R})$-measurable functions for a certain well defined pair $(\Phi, \mathfrak{R})$.

In the remainder of §5, B is used to denote a real M-ring, Ω to denote the set of all real closed maximal ideals M of B, and J to denote the Boolean algebra of idempotents of B. If $M\in\Omega$, then $x(M)$ stands for the image of x under the natural homomorphism of B onto $B-M$. The symbol $x(\cdot)$ represents the real valued function defined on Ω which takes the value $x(M)$ at the point $M\in\Omega$. For each $e\in J$ consider the subset $U(e)=\{M\mid e(M)=1\}$. $\mathfrak{A}$ is used to denote the collection of all such subsets.

LEMMA. *$\mathfrak{A}$ is a σ-algebra and is isomorphic to J.*

PROOF. If M is a closed ideal, then $M\cap J$ is a σ-prime ideal of J and if $\bigcap_\Omega M=0$, then $\bigcap_\Omega M\cap J=0$ also. By a result of Sikorski [**11**, Theorem 1.3], $\mathfrak{A}$ is a σ-algebra and is isomorphic to the σ-complete Boolean algebra J.

Now we may use $R(\Omega, \mathfrak{A})$ to denote the M-ring of all $(\Omega, \mathfrak{A})$-measurable functions and $\overline{R}(\Omega, \mathfrak{A})$ to denote the M-ring of all bounded $(\Omega, \mathfrak{A})$-measurable functions.

THEOREM 7. *B is ring- and lattice-isomorphic to $B(\Omega, \mathfrak{A})$, an M-ring of $(\Omega, \mathfrak{A})$-measurable functions.*

PROOF. First, by the standard (Gelfand [**3**]) argument, B can be shown to be ring-isomorphic to a ring $B(\Omega)$ of real valued functions defined on Ω.

The mapping $x\rightarrow x(M)$ of B onto $B-M$ preserves order. Indeed, suppose $x\geqq 0$ and $x(M)\leqq 0$. Then $x-x(M)\geqq -x(M)\geqq 0$, and, because B is real, $x-x(M)$ has an inverse. However, $x-x(M)$ belongs to M. Therefore $x\geqq 0$ is a sufficient condition for $x(M)\geqq 0$; hence the mapping preserves order.

Define $x(\cdot)\geqq y(\cdot)$ if $x(M)\geqq y(M)$ for each $M\in\Omega$. This definition induces a partial order on $B(\Omega)$, and with $B(\Omega)$ thus ordered, the isomorphism mentioned in the first paragraph of this proof preserves order. Necessarily the lattice structure is preserved as well; hence B is ring- and lattice-isomorphic to $B(\Omega)$.

Finally, each $x(\cdot)\in B(\Omega)$ is a $(\Omega, \mathfrak{A})$-measurable function. Indeed, each $M\in\Omega$ is an l-ideal as well as a real closed maximal ideal. (To see that this is so, look at the image of M in $B(\Omega)$.) From this, it follows that $x\in M$ if and only if $\bar{e}_x\in M$. Therefore

$$\begin{aligned} U[e_{(x-\lambda)^+}] &= \{M \mid e_{(x-\lambda)^+}(M) = 1\} \\ &= \{M \mid (x-\lambda)^+(M) = 0\} \\ &= \{M \mid x(M) \leqq \lambda\} \end{aligned}$$

belongs to $\mathfrak{A}$ for each λ. Thus the symbol $B(\Omega)$ may be meaningfully replaced by the symbol $B(\Omega, \mathfrak{A})$ and the theorem is proved.

It should be remarked at this point that Theorem 7 is still valid when Ω is replaced by any subset Ω^* where the intersection of all ideals in Ω^* is the zero ideal. $\mathfrak{A}$ is defined in the same manner with respect to Ω^* as it was with respect to Ω.

The following theorem is closely related to Theorem 7.

THEOREM 7A. *The following statements are equivalent.* (i) $R=\overline{R}$. (ii) *R is the F-ring of all ordered n-tuples of real numbers for some fixed integer n.* (iii) *All maximal ideals of R are closed.*

PROOF. (i) *implies* (ii). Since $R=\overline{R}$ is a regular real semisimple commutative Banach algebra, it is finite dimensional [1, Theorem 3.5] and it has a representation as a ring of functions. The result follows because the number of maximal ideals of R is then finite.

(ii) *implies* (i) and (ii) *implies* (iii) are trivial.

(iii) *implies* (ii). If all maximal ideals of R are closed, then the mapping $M\rightarrow M\cap\overline{R}$ is a one-to-one correspondence between the maximal ideals of R and those of $\overline{R}$ (Theorems 4, 5, and 6), and in addition each maximal ideal of $\overline{R}$ is closed (Theorem 6).

Since every maximal ideal M of R is closed, it follows easily that $x\in M$ if and only if $e_x\notin M$. Hence if $x\in\overline{R}$, $x+e_x$ belongs to no maximal ideal of $\overline{R}$. Therefore $x+e_x$ has an inverse in $\overline{R}$; so $\overline{R}$ is regular (Theorem 1). By an argument similar to that used in the first paragraph of the proof, $\overline{R}$ is finite dimensional. From the one-to-one correspondence $M\rightarrow M\cap\overline{R}$, we deduce that R is also finite dimensional, and therefore, by Theorem 7, Statement (ii) follows.

DEFINITION. An algebra A of $(\Phi, \mathfrak{L})$-measurable functions is *σ-convex* provided that $y(\cdot)\in A$ if $y(\cdot)$ is $(\Phi, \mathfrak{L})$-measurable and $0\leqq y(\cdot)\leqq x(\cdot)$ where $x(\cdot)$ belongs to A.

Such a σ-convex algebra is necessarily a real M-ring; the following theorem is a converse to this statement.

THEOREM 8. $B(\Omega, \mathfrak{A})$ (*defined in Theorem* 7) *is a σ-convex algebra.*

PROOF. In order to show $B(\Omega, \mathfrak{A})$ is σ-convex, it is first necessary to show $\overline{R}(\Omega, \mathfrak{A})\subseteq B(\Omega, \mathfrak{A})$. It should be noted that since a lattice isomorphism preserves sup's and inf's whenever they occur, the definition of order in $B(\Omega, \mathfrak{A})$ implies that the sup in $B(\Omega, \mathfrak{A})$ is the pointwise sup. Since $B(\Omega, \mathfrak{A})$ contains all simple functions (finite linear combinations of characteristic functions of sets in $\mathfrak{A}$), and since every bounded measurable function is the pointwise sup of a countable set of simple functions, it follows from the conditional σ-completeness of $B(\Omega, \mathfrak{A})$ that $\overline{R}(\Omega, \mathfrak{A})$ is a sub-M-ring of $B(\Omega, \mathfrak{A})$.

To finish the proof, suppose that $y(\cdot)\geqq 0$ is a $(\Omega, \mathfrak{A})$-measurable function and that $y(\cdot)\leqq x(\cdot)\in B(\Omega, \mathfrak{A})$. Let $y_n(\cdot)=y(\cdot)\chi\{M|y(M)$

$\leqq n\}$ for integers $n \geqq 1$. Each element $y_n(\cdot)$ is bounded; hence it belongs to $B(\Omega, \mathfrak{A})$. In addition, $y_n(\cdot) \leqq x(\cdot)$ for each n. Therefore since $B(\Omega, \mathfrak{A})$ is an M-ring, $y(\cdot) = \bigvee_{n=1}^{\infty} y_n(\cdot)$ also belongs to $B(\Omega, \mathfrak{A})$.

COROLLARY. *If B is regular, then* $B(\Omega, \mathfrak{A}) = R(\Omega, \mathfrak{A})$.

PROOF. Suppose $0 \leqq x(\cdot)$ is an $(\Omega, \mathfrak{A})$-measurable function. Then $x(\cdot)+1$ is also $(\Omega, \mathfrak{A})$-measurable and $y(\cdot) = (x(\cdot)+1)^{-1}$ is both measurable and bounded. By Theorem 8, $y(\cdot) \in B(\Omega, \mathfrak{A})$, and since $y(M) \neq 0$ for all $M \in \Omega$, it follows that the characteristic function $\chi\{M \mid y(M)=0\} = 0$ and if y is the isomorphic copy in B of $y(\cdot)$, then the isomorphic copy e_y of the characteristic function $\chi\{M \mid y(M)=0\}$ is zero. Therefore, y^{-1} exists in B, and $y^{-1}(\cdot) = x(\cdot)+1 \in B(\Omega, \mathfrak{A})$. Thus $x(\cdot) \in B(\Omega, \mathfrak{A})$, and the corollary follows from this.

An F-ring A is *atomic* if its Boolean algebra of idempotents is an atomic Boolean algebra. We conclude with two theorems about atomic F-rings.

THEOREM 9. *If R is atomic, then R is an M-ring.*

PROOF. In the course of the proof of Theorem 6, it was shown that there is a one-to-one correspondence $M \rightarrow M \cap I$ between the set of closed maximal ideals of R and the set of σ-prime ideals of I. Sikorski [11, Theorem 1.8] has shown that if a σ-complete Boolean algebra is atomic, then the intersection of its σ-prime ideals is zero. Since $x \in M$ if and only if $\bar{e}_x \in M \cap I$, it follows that

$$\bigcap_{M \in \Omega} M = \bigcap_{M \in \Omega} M \cap I = 0;$$

hence R is an M-ring.

An F-ring A is *complete* if any arbitrary set of elements of A, bounded above by an element of A, has a least upper bound.

THEOREM 10. *If R is both complete and atomic, then it is a direct sum of real fields.*

PROOF. To show that R is a direct sum of real fields, it suffices to show that R is isomorphic to the ring of all real-valued functions on some space Ω^*. Let Ω^* be the set of maximal ideals $M_a = \{x \mid xa = 0\}$ where a is an atom of I. Each ideal M_a is closed and since $\bigcap_{\Omega^*} M_a \cap I = 0$, we deduce that $\bigcap_{\Omega^*} M_a = 0$ also (see proof of Theorem 10).

By the remark following the proof of Theorem 7, R is isomorphic to $R(\Omega^*, \mathfrak{A})$ ($\mathfrak{A}$ is defined relative to I and, by Lemma, is isomorphic to I). The complete atomic character of R insures that $\mathfrak{A}$ is the Boolean algebra of all subsets of Ω^*. Hence $R(\Omega^*, \mathfrak{A})$ is the M-ring of all real functions on Ω^*.

References

1. R. F. Arens and I. Kaplansky, *Topological representations of algebras*, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 457–481.

2. G. Birkhoff, *Lattice theory*, New York, Amer. Math. Soc. Colloquium Publications, vol. 25, 1948.

3. I. Gelfand, *Normierte Ringe*, Rec. Math. (Mat. Sbornik) N.S. vol. 9 (51) (1941) pp. 3–24.

4. L. Gillman and M. Henriksen, *Concerning rings of continuous functions*, Trans. Amer. Math. Soc. vol. 77 (1954) pp. 340–362.

5. L. Gillman, M. Henriksen, and M. Jerison, *On a theorem of Gelfand and Kolmogorof concerning maximal ideals in rings of continuous functions*. Proc. Amer. Math. Soc. vol. 5 (1954) pp. 447–455.

6. E. Hewitt, *Rings of real-valued continuous functions* I, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 45–99.

7. D. R. Morrison, *Bi-regular rings and the ideal lattice isomorphisms*, Proc. Amer. Math. Soc. vol. 6 (1955) pp. 46–49.

8. S. Kakutani, *Concrete representation of abstract L-spaces and the mean ergodic theorem*, Ann. of Math. vol. 42 (1941) pp. 523–537.

9. H. Nakano, *Modern spectral theory*, Tokyo Mathematical Book, Series 2, Tokyo, 1950.

10. J. von Neumann, *Regular rings*, Proc. Nat. Acad. Sci. U.S.A. vol. 22 (1936) pp. 707–713.

11. R. Sikorski, *On the representation of Boolean algebras as fields of sets*, Fund. Math. vol. 35 (1948) pp. 247–258.

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