

HIGHER DIMENSIONAL INDECOMPOSABLE CONNECTED SETS

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Here we wish to show the existence of indecomposable, nonclosed, connected sets, especially hereditary ones, contained densely in connected domains of E_n , $n > 1$, or in the Hilbert cube I_ω . We recall that: an *indecomposable connected set* M is a connected set which is not the sum of two connected subsets, each with a different closure than that of M ; it is also *hereditary* if every connected subset is an indecomposable connected set. We call a nondegenerate connected set a *connexe*. A closed indecomposable connexe is an indecomposable continuum.

We have shown with Hunter in [4] that there exists a hereditarily indecomposable connexe contained densely in a connected domain of E_2 . Also it was shown in Theorem 4 of [4] that there exists in a solid torus of E_3 an arc-wise path, without any crossing, densely contained interior to the torus, which is an indecomposable connexe: it is the proof of this Theorem 4 we wish to modify to give the results of this paper; by an arc-wise path P here we mean a set of points P which is the sum of simple continuous arcs $p_i p_{i+1}$ ($i=0, 1, 2, \dots$), p_i a point, such that $p_0 p_1 + p_1 p_2 + \dots + p_{h-1} p_h$ is a simple continuous arc, for each h , from p_0 to p_h . Thus intuitively we may think of the indecomposable connexe P as generated by an idealized particle Q moving over the points of P , being at time t_i at point p_i where $\text{Lim } t_i = \infty$ say.

To give a quicker intuitive grasp of the methods used below we wish to speak of a *basic connexe densely extendable* over a connected domain, which may be the interior of a torus in E_3 or its generalization in E_n or I_ω , to give a desired indecomposable connexe. Such a connexe, for Theorem 4 of [4] is an arc minus one end point: for if both this and the arc-wise path P are considered as 1-dimensional spaces in themselves with a 1-dimensional topology, i.e. region is an arc minus end points in both, they are homeomorphic. The construction below is dependent upon an indecomposable connexe M with only one composant, which is r -dimensional at each of its points, and this connexe is contained densely in a domain D . If we take a basis of connected, r -dimensional regions for the topology of this composant, then the basic connexe densely extendable over D for M is a set W which is homeomorphic with this composant under a similar

Received by the editors December 18, 1956.

topology, i.e. with a basis of connected r -dimensional regions. See [3, pp. 2-3, 70]; for fundamental definitions see [1; 2; or 3]. For E_3 below we use a solid triangle minus one vertex for W , and for higher dimensional spaces W will be the r -dimensional generalization of this, i.e. a solid r -triangle, or r -pyramid, minus one vertex A . This *solid r -pyramid* will be homeomorphic to an r -dimensional solid with center line a straight line BA and with cross sections perpendicular to this consisting of the cartesian product of $(r-1)$ straight line intervals, where, if d_x is the diameter of this cross section at x on BA and if x approaches the limit A , then $\text{Lim } d_x = 0$: the solid r -pyramid is the sum of these cross sections for $x \in BA$, but $x \neq A$. Intuitively we may think of generating this solid r -pyramid by letting x move from B to A on BA carrying this cross section with it. We will call the solid triangle minus a vertex a solid 2-pyramid. By the *vertex* of an r -pyramid we will mean this missing vertex A .

THEOREM 1. *Let D be a connected domain imbedded in E_n , $n > 2$, or I_ω and let W be a solid r -pyramid. Then there exists an indecomposable connexe M , with only one composant and it r -dimensional, densely contained in D whose basic connexe densely extendable over D is W .*

PROOF. We give the proof first for E_3 and $r=2$, observing however that the fundamentals used in the proof also have equations in the higher dimensional spaces. Let E_3 be an $(x_1x_2x_3)$ -coordinate space. We obtain this proof as a modification of that of Theorems 1 and 4 of [4]: this in turn is obtained by combining the tunneling process of Wada as used in [5, pp. 178-179, Theorem 1] for the construction of an indecomposable connexe, with the method of constructing a simple continuous arc as in [2, pp. 86-88, Theorem 1] or in [3, p. 80, Theorem 3.9]. In [4] we obtained our covering of D , which gives the simple chain of regions needed for the construction of the arc, by taking interiors of spheres as regions: here we will take instead the interiors of ellipsoids, with larger axis parallel to the x_1 -axis: with center at $(0, 0, 0)$ the equation of the ellipsoid will be taken as $(x_1/a_1)^2 + (x_2/a_2)^2 + (x_3/a_3)^2 = 1$, where $a_1 > a_2$ or a_3 .

In the proof in [4] a set of points $p_0, p_1, \dots, p_j, \dots$ is taken in D in such a manner that one can have the construction below. A simple chain of regions, C_1 , is taken from p_0 to p_1 ; a simple chain C_2 is taken which is the sum of chains from p_0 to p_1 and from p_1 to p_2 ; in general a simple chain C_i is taken from p_0 to p_i , which is the sum of simple subchains from p_0 to p_1 , from p_1 to p_2 , \dots , and from p_{i-1} to p_i . Let T_i be the sum of the regions of C_i ; let Z_i be the boundary of T_i except for the part of this boundary which is also part of the boundary

of the first and last regions of C_i : since Z_i is homeomorphic to a cylinder, we will call it a cylinder. The chains C_i are taken so that the Z_i ($i=1, 2, \dots$) are mutually exclusive; also the parts of T_i and T_{i+1} from p_{h-1} to p_h , $1 \leq h \leq i$, i.e. the sums of the regions of these subchains of C_i and C_{i+1} , T'_i and T'_{i+1} say, are such that $T'_i \supset \bar{T}'_{i+1}$. Furthermore these chains are taken so that $\cap \bar{T}'_i$ is a simple continuous arc $p_{h-1}p_h$. Thus, if r_{hi} is the radius of the largest sphere of the chain C_i from p_{h-1} to p_h , $\text{Lim } r_{hi} = 0$. Also the chains are taken so that $p_0p_1 + p_1p_2 + \dots + p_{g-1}p_g$ ($g=1, 2, \dots$) is an arc; and $M = p_0p_1 + p_1p_2 + \dots + p_{g-1}p_g + \dots$ is dense in D . All of this can be done by well known methods.

We see that M is an indecomposable connexe by the usual method: Suppose M is the sum of the two connected subsets H and K neither of which has the same closure as M does. Let R_h and R_k be regions such that $H \supset \bar{R}_h \cdot M$, $\bar{R}_h \cdot K = 0$, and $K \supset \bar{R}_k \cdot M$. By the method of construction giving the above properties there exists a simple subchain of some C_i whose end regions are in R_h and one of whose regions is contained in R_k , but does not contain all of $R_k \cdot M$. Thus a subcontinuum of (Z_i plus the boundary of R_h) separates the connexe K , which is a contradiction, since neither Z_i nor the boundary of R_h contains a point of K .

The proof of Theorem 1 here is similar, taking ellipsoids of revolution with large axis parallel to the x_1 -axis of the space, in place of the spheres in [4]: thus in [4] the cylinders Z_i have circular cross sections, while here these cross sections are ellipses. If a_{hi} is the larger semiaxis, r_{hi} the smaller for the largest of these cross sections, and b_{hi} is the largest semiaxis of the smallest of the cross sections of Z_i between p_{h-1} and p_h , then we take $\text{Lim } r_{hi}(i \rightarrow \infty) = 0$, $\text{Lim } a_{hi}(i \rightarrow \infty) = a_h$, and $\text{Lim } b_{hi}(i \rightarrow \infty) \neq 0$. Each ellipsoid with smaller semiaxis r_i has about its center a sphere of radius r_i . If these spheres are substituted for the ellipsoids, we may take the chains so as to get arcs as above. In order that the indecomposable connexe M of Theorem 1 be dense in D we must take a_h above so that $\text{Lim } a_h(h \rightarrow \infty) = 0$. Thus, in part by the above methods, we see without great difficulty that this Theorem is true for E_3 and $r=2$. For $r > 2$ in E_n we must treat $r-1$ of the semiaxes as we did a_{hi} above and the remaining as r_{hi} was treated: thus for this case we use generalized ellipsoids rather than ellipsoids of revolution. Similar modifications indicate the Theorem is true for I_ω , in spite of the intuitive nature of the argument above.

COROLLARY 1.1. *Let D be a connected domain imbedded in E_n , $n > 2$, or in I_ω . Then D contains densely an indecomposable connexe M with*

an uncountable number of composants, each of which is an arc-wise path dense in D .

PROOF. This follows from Theorem 1: for if W is the solid r -pyramid with base W' and vertex A , one can join uncountably many of the points of W' to A by straight line intervals. With the obvious r -dimensional topology on both W and the composant of M in Theorem 1, we have a homeomorphism between these which carries these straight line intervals into the desired composants of this corollary.

COROLLARY 1.2. *Let D be a connected domain imbedded in E_n , $n > 1$, or in I_ω . Then D contains densely a hereditarily indecomposable connexe M of r -dimensions, $1 \leq r < \text{the dimension of the space}$.*

PROOF. For $n = 2$ this is Theorem 9 of [4]. Thus let the space be as in Theorem 1 above, from which this corollary follows. Let W be as in the proof of Corollary 1.1 above. Let I be a hereditarily indecomposable continuum of r -dimensions of [6], or of [7] if $r = 2$, such that I is imbedded in $W + A$ of the proof above and I contains both A and points of the base W' . Thus the homeomorphism as above which carries W into the one composant of Theorem 1, carries I into the set of points M of this corollary, which is the desired hereditarily indecomposable connexe densely contained in D . We note that, if C is a connexe of M such that $\bar{C} \neq \bar{M}$, then \bar{C} is a proper subcontinuum of the one composant of the indecomposable connexe of Theorem 1; hence here C must be indecomposable. Thus the proof gives no difficulty.

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