HIGHER DIMENSIONAL INDECOMPOSABLE CONNECTED SETS

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Here we wish to show the existence of indecomposable, nonclosed, connected sets, especially hereditary ones, contained densely in connected domains of E_n, $n>1$, or in the Hilbert cube I_ω. We recall that: an *indecomposable connected set* M is a connected set which is not the sum of two connected subsets, each with a different closure than that of M; it is also *hereditary* if every connected subset is an indecomposable connected set. We call a nondegenerate connected set a *connexe*. A closed indecomposable connexe is an indecomposable continuum.

We have shown with Hunter in [4] that there exists a hereditarily indecomposable connexe contained densely in a connected domain of E_2. Also it was shown in Theorem 4 of [4] that there exists in a solid torus of E_3 an arc-wise path, without any crossing, densely contained interior to the torus, which is an indecomposable connexe: it is the proof of this Theorem 4 we wish to modify to give the results of this paper; by an arc-wise path P here we mean a set of points P which is the sum of simple continuous arcs p_ip_{i+1} $(i=0, 1, 2, \cdots)$, p_i a point, such that $p_0p_1+p_1p_2+\cdots+p_{h-1}p_h$ is a simple continuous arc, for each h, from p_0 to p_h. Thus intuitively we may think of the indecomposable connexe P as generated by an idealized particle Q moving over the points of P, being at time t_i at point p_i where $\operatorname{Lim} t_i=\infty$ say.

To give a quicker intuitive grasp of the methods used below we wish to speak of a *basic connexe densely extendable* over a connected domain, which may be the interior of a torus in E_3 or its generalization in E_n or I_ω, to give a desired indecomposable connexe. Such a connexe, for Theorem 4 of [4] is an arc minus one end point: for if both this and the arc-wise path P are considered as 1-dimensional spaces in themselves with a 1-dimensional topology, i.e. region is an arc minus end points in both, they are homeomorphic. The construction below is dependent upon an indecomposable connexe M with only one composant, which is r-dimensional at each of its points, and this connexe is contained densely in a domain D. If we take a basis of connected, r-dimensional regions for the topology of this composant, then the basic connexe densely extendable over D for M is a set W which is homeomorphic with this composant under a similar

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topology, i.e. with a basis of connected r-dimensional regions. See [**3**, pp. 2–3, 70]; for fundamental definitions see [**1**; **2**; or **3**]. For E_3 below we use a solid triangle minus one vertex for W, and for higher dimensional spaces W will be the r-dimensional generalization of this, i.e. a solid r-triangle, or r-pyramid, minus one vertex A. This *solid r-pyramid* will be homeomorphic to an r-dimensional solid with center line a straight line BA and with cross sections perpendicular to this consisting of the cartesian product of $(r-1)$ straight line intervals, where, if d_x is the diameter of this cross section at x on BA and if x approaches the limit A, then $\operatorname{Lim} d_x=0$: the solid r-pyramid is the sum of these cross sections for $x\in BA$, but $x\neq A$. Intuitively we may think of generating this solid r-pyramid by letting x move from B to A on BA carrying this cross section with it. We will call the solid triangle minus a vertex a solid 2-pyramid. By the *vertex* of an r-pyramid we will mean this missing vertex A.

THEOREM 1. *Let D be a connected domain imbedded in E_n, $n>2$, or I_ω and let W be a solid r-pyramid. Then there exists an indecomposable connexe M, with only one composant and it r-dimensional, densely contained in D whose basic connexe densely extendable over D is W.*

PROOF. We give the proof first for E_3 and $r=2$, observing however that the fundamentals used in the proof also have equations in the higher dimensional spaces. Let E_3 be an $(x_1x_2x_3)$-coordinate space. We obtain this proof as a modification of that of Theorems 1 and 4 of [**4**]: this in turn is obtained by combining the tunneling process of Wada as used in [**5**, pp. 178–179, Theorem **1**] for the construction of an indecomposable connexe, with the method of constructing a simple continuous arc as in [**2**, pp. 86–88, Theorem **1**] or in [**3**, p. 80, Theorem 3.9]. In [**4**] we obtained our covering of D, which gives the simple chain of regions needed for the construction of the arc, by taking interiors of spheres as regions: here we will take instead the interiors of ellipsoids, with larger axis parallel to the x_1-axis: with center at (0, 0, 0) the equation of the ellipsoid will be taken as $(x_1/a_1)^2+(x_2/a_2)^2+(x_3/a_3)^2=1$, where $a_1>a_2$ or a_3.

In the proof in [**4**] a set of points $p_0, p_1, \cdots, p_j, \cdots$ is taken in D in such a manner that one can have the construction below. A simple chain of regions, C_1, is taken from p_0 to p_1; a simple chain C_2 is taken which is the sum of chains from p_0 to p_1 and from p_1 to p_2; in general a simple chain C_i is taken from p_0 to p_i, which is the sum of simple subchains from p_0 to p_1, from p_1 to p_2, $\cdots$, and from p_{i-1} to p_i. Let T_i be the sum of the regions of C_i; let Z_i be the boundary of T_i except for the part of this boundary which is also part of the boundary of the first and last regions of C_i: since Z_i is homeomorphic to a cylinder, we will call it a cylinder. The chains C_i are taken so that the Z_i ($i=1, 2, \cdots$) are mutually exclusive; also the parts of T_i and T_{i+1} from p_{h-1} to p_h, $1 \leqq h \leqq i$, i.e. the sums of the regions of these subchains of C_i and C_{i+1}, T_i' and T_{i+1}' say, are such that $T_i' \supset \overline{T}_{i+1}'$. Furthermore these chains are taken so that $\bigcap \overline{T}_i'$ is a simple continuous arc $p_{h-1}p_h$. Thus, if r_{hi} is the radius of the largest sphere of the chain C_i from p_{h-1} to p_h, $\operatorname{Lim} r_{hi}=0$. Also the chains are taken so that $p_0p_1+p_1p_2+ \cdots +p_{g-1}p_g$ ($g=1, 2, \cdots$) is an arc; and $M = p_0p_1+p_1p_2+ \cdots +p_{g-1}p_g+ \cdots$ is dense in D. All of this can be done by well known methods.

We see that M is an indecomposable connexe by the usual method: Suppose M is the sum of the two connected subsets H and K neither of which has the same closure as M does. Let R_h and R_k be regions such that $H \supset \overline{R}_h \cdot M$, $\overline{R}_h \cdot K=0$, and $K \supset \overline{R}_k \cdot M$. By the method of construction giving the above properties there exists a simple subchain of some C_i whose end regions are in R_h and one of whose regions is contained in R_k, but does not contain all of $R_k \cdot M$. Thus a subcontinuum of (Z_i plus the boundary of R_h) separates the connexe K, which is a contradiction, since neither Z_i nor the boundary of R_h contains a point of K.

The proof of Theorem 1 here is similar, taking ellipsoids of revolution with large axis parallel to the x_1-axis of the space, in place of the spheres in [4]: thus in [4] the cylinders Z_i have circular cross sections, while here these cross sections are ellipses. If a_{hi} is the larger semiaxis, r_{hi} the smaller for the largest of these cross sections, and b_{hi} is the largest semiaxis of the smallest of the cross sections of Z_i between p_{h-1} and p_h, then we take $\operatorname{Lim} r_{hi}(i\rightarrow\infty)=0$, $\operatorname{Lim} a_{hi}(i\rightarrow\infty) =a_h$, and $\operatorname{Lim} b_{hi}(i\rightarrow\infty)\neq 0$. Each ellipsoid with smaller semiaxis r_i has about its center a sphere of radius r_i. If these spheres are substituted for the ellipsoids, we may take the chains so as to get arcs as above. In order that the indecomposable connexe M of Theorem 1 be dense in D we must take a_h above so that $\operatorname{Lim} a_h(h\rightarrow\infty)=0$. Thus, in part by the above methods, we see without great difficulty that this Theorem is true for E_3 and $r=2$. For $r>2$ in E_n we must treat $r-1$ of the semiaxes as we did a_{hi} above and the remaining as r_{hi} was treated: thus for this case we use generalized ellipsoids rather than ellipsoids of revolution. Similar modifications indicate the Theorem is true for I_ω, in spite of the intuitive nature of the argument above.

COROLLARY 1.1. *Let D be a connected domain imbedded in E_n, $n>2$, or in I_ω. Then D contains densely an indecomposable connexe M with* *an uncountable number of composants, each of which is an arc-wise path dense in D.*

PROOF. This follows from Theorem 1: for if W is the solid r-pyramid with base W' and vertex A, one can join uncountably many of the points of W' to A by straight line intervals. With the obvious r-dimensional topology on both W and the composant of M in Theorem 1, we have a homeomorphism between these which carries these straight line intervals into the desired composants of this corollary.

COROLLARY 1.2. *Let D be a connected domain imbedded in E_n, $n>1$, or in I_ω. Then D contains densely a hereditarily indecomposable connexe M of r-dimensions, $1 \leqq r <$ the dimension of the space.*

PROOF. For $n=2$ this is Theorem 9 of [4]. Thus let the space be as in Theorem 1 above, from which this corollary follows. Let W be as in the proof of Corollary 1.1 above. Let I be a hereditarily indecomposable continuum of r-dimensions of [6], or of [7] if $r=2$, such that I is imbedded in $W+A$ of the proof above and I contains both A and points of the base W'. Thus the homeomorphism as above which carries W into the one composant of Theorem 1, carries I into the set of points M of this corollary, which is the desired hereditarily indecomposable connexe densely contained in D. We note that, if C is a connexe of M such that $\overline{C} \neq \overline{M}$, then $\overline{C}$ is a proper subcontinuum of the one composant of the indecomposable connexe of Theorem 1; hence here C must be indecomposable. Thus the proof gives no difficulty.

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