

## ON CANONICAL CONSTRUCTIONS IV

MURRAY GERSTENHABER

In the second paper in this series [2; 3], it was demonstrated that under suitable topological conditions on a space  $S$  (which are in particular always satisfied for a manifold), the problem of reconstructing  $S$  from its group  $G$  of homeomorphism onto itself may be reduced to the problem of reconstructing the points of  $S$  from  $G$ . If  $S$  can be reconstructed then every automorphism of  $G$  is inner, and it was shown in this way that every automorphism of  $G$  is in fact inner when  $S$  is the unit interval; this is the theorem of Fine and Schweigert [1]. The present paper is devoted principally to proving the Fine-Schweigert theorem for the disc. Given a space  $S$  topologically equivalent to the interior of the unit circle in the plane, the point set of  $S$  will be canonically reconstructed from the group  $G$  of homeomorphisms of  $S$  onto itself. A principal tool is the classical theorem that if  $g$  is an orientation-preserving homeomorphism of finite order of the disc  $S$  onto itself, then  $S$  may be mapped homeomorphically on the interior of the unit circle in such a way that  $g$  becomes a rotation, [4]. Since a homeomorphism of odd order is necessarily orientation-preserving, it follows that an element of odd order (different from one) of the group  $G$  of homeomorphisms of  $S$  onto itself has precisely one fixed point, which will be called its center. If  $x$  is any point of  $S$ , let  $x'$  denote the set of all elements of  $G$  of finite odd order (different from one) having  $x$  for center. The collection  $S'$  of all sets  $x'$  is in canonical one-one correspondence with the point set of  $S$ . Our problem will be at an end if  $S'$  can be exhibited as a structure derived from the purely algebraic structure  $G$ . To this end it is sufficient to be able to decide within the structure of  $G$  when two elements of odd order (different from one) have distinct centers. This occupies the remainder of the paper. Henceforth, when an element of  $G$  is said to be of odd order, it will be understood that the order is different from one.

**Criterion that two elements of  $G$  of odd order have the same center.** Let  $D$  be the interior of the unit circle in the plane,  $R$  the set of rotations of odd order of  $D$ , and  $R^*$  the group generated by all homeomorphisms of  $D$  onto itself which commute with some element

---

Received by the editors May 27, 1956 and, in revised form, December 19, 1956.

of  $R$ . Since every element of  $R$  has exactly one fixed point, the center 0 of  $D$ , it follows that every generator of  $R^*$  leaves 0 fixed, and every element of  $R^*$  therefore does likewise.

**LEMMA.** *Let  $p$  be any point of  $D$  other than the center 0. Then there exists a neighborhood  $N$  of  $p$  such that if  $h$  is a homeomorphism of  $D$  onto itself which is the identity outside  $N$  then  $h$  is in  $R^*$ .*

**PROOF.** We may without loss of generality assume that  $p$  lies on the positive  $x$ -axis. Let  $r$  denote a rotation of  $D$  counterclockwise through an angle of  $2\pi/n$ , where  $n$  is some positive integer greater than one. If  $h$  is a homeomorphism of  $D$  onto itself whose carrier  $c(h)$  subtends an angle of less than  $2\pi/n$  at 0, then the images of  $c(h)$  under  $r$  are disjoint and the homeomorphisms  $r^i h r^{-i}$ ,  $i = 0, 1, \dots, n-1$  all commute. Their product is a homeomorphism which commutes with  $r$  and will be referred to as " $h$  repeated with period  $n$ " or  $h^{(n)}$ . We will consider  $h^{(n)}$  as consisting of  $n$  parts or "repetitions of  $h$ ," the  $i$ th being  $r^i h r^{-i}$ . If  $n$  is divisible by some odd integer  $k$  greater than one, then  $h$  repeated with period  $n$  is in  $R^*$ , for it commutes with the rotations of  $D$  of order  $k$ .

Consider now the rays through the center 0 of  $D$  with arguments  $2m\pi/5$  and  $2n\pi/11$ ,  $m$  and  $n$  integers. One may observe geometrically that there exists a homeomorphism  $s$  of  $D$  onto itself such that

1.  $s$  commutes with the rotation of  $D$  through  $2\pi/3$  (whence  $s$  is in  $R^*$ ),
2.  $s$  carries each ray with argument  $2m\pi/5$  into the nearest one with argument of the form  $2n\pi/11$  and
3. there is an  $\epsilon > 0$  such that in the region between the rays  $(2m\pi/5) - \epsilon$  and  $(2m\pi/5) + \epsilon$ ,  $s$  behaves like a rotation.

If now  $h$  is a homeomorphism of  $D$  onto itself whose carrier subtends an angle at 0 less than the minimum of  $2\pi/11$  and  $\epsilon$  then we may construct  $h^{(6)}$ ,  $h^{(11)}$ , and observe that in  $(sh^{(6)}s^{-1})^{-1}h^{(11)}$  five of the repetitions of  $h$  in  $h^{(11)}$  have been cancelled out. We may cancel out five more of the repetitions of  $h$  by replacing  $s$  by its conjugate under a rotation through  $2\pi/11$ . In this fashion we can cancel all but one of the repetitions and obtain therefore a conjugate of  $h$  under some power of the rotation through  $2\pi/11$  represented as a product of elements of  $R^*$ . It follows that  $h$  is in  $R^*$ . Therefore, taking the neighborhood  $N$  to subtend an angle at 0 less than the minimum of  $2\pi/11$  and  $\epsilon$ , the lemma is proved.

Let  $S$  be as before a space homeomorphic to the interior of the unit disc and  $f$  a homeomorphism of odd order. Let  $R(f)$  denote the set of all homeomorphisms of odd order which commute with  $f$  and  $R^*(f)$

denote the group generated by all homeomorphisms  $h$  such that  $h$  commutes with at least one of the elements of  $R(f)$ . The homeomorphism  $f$  has exactly one fixed point  $0_f$ , and every element of  $R(f)$  has therefore also exactly one fixed point, the same point. Every generator of  $R^*(f)$  must then leave  $0_f$  fixed, so the same is true of every element of  $R^*(f)$ .

$S$  may be mapped on the interior of the unit disc  $D$  in such a way that  $f$  becomes a rotation;  $0_f$  is then mapped on the center  $0$ .  $R(f)$  then contains  $R$  and  $R^*(f)$  therefore also contains  $R^*$ . From the preceding lemma it follows that if  $p$  is any point distinct from  $0_f$ , then there exists a neighborhood  $N$  of  $p$  such that if  $h$  is any homeomorphism whose carrier is contained in  $N$  then  $h$  is in  $R^*(f)$ .

Let  $f$  and  $g$  be two homeomorphisms of odd order. We seek an algebraic criterion which will tell when the centers of  $f$  and  $g$  are identical. Let  $R^*(f, g)$  denote the group generated by  $R^*(f)$  and  $R^*(g)$ . If the centers of  $f$  and  $g$  are identical then every element of this group leaves  $0_f$  fixed. It follows that the intersection of all the conjugates of  $R^*(f, g)$  contains only the identity (an algebraic statement). On the other hand, if the centers of  $f$  and  $g$  are distinct then  $R^*(f, g)$  has the property that if  $p$  is any point of  $S$  (without exception) then there exists a neighborhood  $N$  of  $p$  such that if  $h$  is a homeomorphism whose carrier is contained in  $N$  then  $h$  is in  $R^*(f, g)$ ; this is in fact already true of the union of the groups  $R^*(f)$  and  $R^*(g)$ . A group of homeomorphisms with this property will be called *pervasive*. The proof of the main theorem of this paper will be at an end when it has been demonstrated that if  $P$  is any pervasive subgroup of the group of homeomorphisms of  $S$  onto itself then the intersection of all the conjugates of  $P$  is not reduced to the identity, for this will enable us to determine algebraically when  $f$  and  $g$  have the same fixed point. Since any conjugate of a pervasive group is again pervasive, it is sufficient to exhibit a specific homeomorphism  $t$  not the identity of  $S$  onto itself, such that if  $P$  is any pervasive group then  $t$  is in  $P$ . For convenience, an open subset  $N$  of  $S$  with the property that if  $h$  is any homeomorphism whose carrier is contained in  $N$  then  $h$  is in  $P$  will be called a  $P$ -neighborhood. The definition of a pervasive group is that every point of  $S$  has a  $P$ -neighborhood.

We may assume again that  $S$  is the interior of the unit disc  $D$  with polar coordinates  $(r, \theta)$ . Let  $\phi(r)$  be a real-valued continuous function on  $[0, 1]$  such that  $\phi(r) = 0$  for  $r \leq 1/3$  and  $r \geq 2/3$  but such that  $\phi$  is not identically zero. Then  $(r, \theta) \rightarrow (r, \theta + \phi(r))$  defines a homeomorphism  $t$  which we shall show is in every pervasive group  $P$ . (Henceforth we shall write simply  $\theta \rightarrow \theta + \phi(r)$  for such a homeomor-

phism.) The carrier of  $t$  is contained in the interior of the disc of radius  $2/3$  about the center of  $D$ . The closure of this set is compact and may therefore be covered by a finite number of  $P$ -neighborhoods  $N_1, \dots, N_k$ . We may assume these  $P$ -neighborhoods each to be a region bounded between two rays through 0 and two concentric circles with center at 0.

Given any circle  $r=c$  we can choose a subset, say  $N_1, \dots, N_m$  of the  $N_i$  covering it such that each  $N_i$  meets only  $N_{i-1}$  and  $N_{i+1}$ ,  $i=1, \dots, m$ , where the indices are taken modulo  $m$ . There then exists an  $\epsilon > 0$  such that the annulus between  $r=c-\epsilon$  and  $r=c+\epsilon$  is covered by  $N_1 \cup \dots \cup N_m$ . Letting such an annulus be called a  $P$ -annulus, it is the case that the carrier of  $t$  can be covered by a finite number of  $P$ -annuli  $A_1, \dots, A_n$ , where  $A_i$  is the annulus between say,  $r=c_i-\epsilon_i$  and  $r=c_i+\epsilon_i$ . Let  $\psi_1, \dots, \psi_n$  be continuous functions of  $r$  defined on  $[0, 1]$  such that  $\psi_i(r)=0$  for  $r \leq c_i-\epsilon_i$  and  $r \geq c_i+\epsilon_i$  and such that  $\sum \psi_i = 1$ . Let homeomorphisms  $t_i$  be defined by  $\theta \rightarrow \theta + \psi_i(r)\phi(r)$ . Then  $t = t_1 t_2 \dots t_n$ . To prove that  $t$  is in  $P$  it is therefore sufficient to prove that  $t_i$  is in  $P$  for every  $i$ . That is, it is sufficient to show that a homeomorphism  $u$  of the form  $\theta \rightarrow \theta + \lambda(r)$  whose carrier is contained in an annulus  $A$  covered by a finite set of  $P$ -neighborhoods  $N_1, \dots, N_m$  of the special form described and overlapping only in pairs is in  $P$ .

Now we may assume that the  $N_1, \dots, N_m$  all have their two circular boundaries on the same two circles  $r=c$  and  $r=c'$  and that the rectilinear boundaries of  $N_i$  are on the rays  $\theta=\theta_i$  and  $\theta=\theta'_i$ . Choose rays  $\tau_i$  and  $\tau'_i$  such that the sets  $M_i$ ,  $i=1, \dots, m$  bounded by them and the circles  $r=c$  and  $r=c'$  have the properties:

1. The sets  $\overline{M}_i$  are disjoint,
2.  $\overline{M}_i \subset N_i$  and,
3.  $N_i - M_i$  is contained in the union of  $N_{i-1} \cap N_i$  and  $N_i \cap N_{i+1}$ , where the indices may if necessary be taken modulo  $m$ . Note that the sets  $N_i \cap N_{i+1}$  are by assumption mutually disjoint.

Let the homeomorphism  $u^\alpha$  for any real  $\alpha$  be defined by  $\theta \rightarrow \theta + \alpha\lambda(r)$ . To show that  $u$  is in  $P$  it is sufficient to show that  $u^\alpha$  is in  $P$  for all sufficiently small  $\alpha$ . We may in particular choose  $\alpha$  so small that  $u^\alpha(\overline{M}_i)$  is disjoint from  $\overline{M}_j$  for  $j \neq i$  and  $u^\alpha(\overline{M}_i) \subset N_i$ . Then there exist homeomorphisms  $u_i^\alpha$  such that:

1.  $u_i^\alpha$  coincides with  $u^\alpha$  on  $M_i$ ,
2. The carrier of  $u_i^\alpha$  is contained in  $N_i$  and
3. The carriers of the  $u_i^\alpha$  are disjoint. By the second condition each  $u_i^\alpha$  is in the pervasive group  $P$ , whence to show that  $u^\alpha$  is in  $P$  it is sufficient to show that  $v = u^\alpha(u_1^\alpha \cdots u_m^\alpha)^{-1}$  is in  $P$ . The carrier of  $v$  is by the

construction contained in the union of all the  $N_i \cap N_{i+1}$ ,  $i = 1, \dots, m$  modulo  $m$ . These sets are disjoint and each must be carried into itself by  $v$ . Therefore  $v$  is a product of commuting homeomorphisms each of whose carriers is contained in a  $P$ -neighborhood. It is therefore in  $P$  and the proof is at an end. We may finally conclude that every automorphism of the group of homeomorphisms of the disc onto itself is inner.

**Appendix. The automorphisms of the real numbers.** While on the topic of canonical constructions it may be of interest to give an example of a theorem whose proof becomes transparent when considered from the present point of view. The field of real numbers has no automorphism other than the identity.

The real numbers constitute a structure derived from the structure of the positive integers, and the latter structure is further canonically reconstructable from the former, being the semigroup generated by the multiplicative unit under addition. It follows that every automorphism of the real numbers is inner, i.e., induced by an automorphism of the positive integers, but the latter structure has no automorphism other than the identity.

Let it be noted that the same argument does *not* prove that there exists no nontrivial automorphism of the complex numbers. For it is not possible to construct explicitly the complex numbers from the real numbers (and hence from the integers) without explicitly distinguishing between the square roots of  $-1$ . [In the usual construction by pairs of real numbers, for example, we may point to the “positive” root, i.e., the one represented by  $(0, 1)$ .] Clearly no automorphism exists other than the trivial one which preserves this stronger structure. On the other hand, if one means by “the complex numbers” a field algebraically isomorphic to the one obtained by explicit construction, then these “complex numbers” are no longer *constructed* from the reals and indeed have automorphisms. Similar paradoxes are not hard to find; all show that one must treat the notion of a derived structure with caution.

#### REFERENCES

1. N. J. Fine and G. E. Schweigert, *On the group of homeomorphisms of an arc*, Ann. of Math. vol. 62, no. 2 (1955).
2. Murray Gerstenhaber, *On canonical constructions I*, Proc. Nat. Acad. Sci. U.S.A. vol. 41, no. 4 (1955).
3. ———, *On canonical constructions II*, Proc. Nat. Acad. Sci. U.S.A. vol. 42, no. 11 (1956).
4. B. v. Kerekjarto, *Vorlesungen über Topologie I*, Berlin, Springer, 1923, p. 224.