NORMAL SUBGROUPS OF MONOMIAL GROUPS

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1. Introduction. Let $U$ be a set, $o(U) = B = \aleph_u$, $u \geq 0$, where $o(U)$ means the number of elements of $U$. Let $H$ be a fixed group. A monomial substitution $y$ is a transformation that maps every $x$ of $U$ in a one-to-one fashion into an $x$ of $U$ multiplied by an element $h_x$ of $H$. Multiplication of substitutions means successive applications. The set of all monomial substitutions forms a group $\Sigma$. Ore [1] has studied this group for finite $U$, and some of his results have been generalized to general $U$ in [3].

The normal subgroups for one subgroup of $\Sigma$ have been determined [3]. This paper extends those results to the extent of determining the structure of all normal subgroups of various subgroups of $\Sigma$ in a rather general case. These results are stated in Theorems 1, 2, 3, 4, 5, 6.

2. Definitions. Let $d$ be the cardinal of the integers. Let $B$ be an infinite cardinal; $B^+$, the successor of $B$; $U$, a set such that $o(U) = B$; and $C$ such that $d < C \leq B^+$. Let $H$ be a fixed group and $e$ the identity of $H$. Denote by $\Sigma = \Sigma(H, B, d, C)$ the monomial group of $U$ over $H$ whose elements are of the form

$$y = \left( \ldots, x_e, \ldots, h_x x_i, \ldots \right)$$

where only a finite number of the $h_x$ are not $e$ and the number of $x$ not mapped into themselves is less than $C$. Any element of $\Sigma$ may be written in the form

$$y = \left( \ldots, x_e, \ldots \right) \left( \ldots, h_x x_i, \ldots \right) \left( \ldots, e x_i, \ldots \right)$$

or $y = vs$ where $v$ sends every $x$ into itself, every $h$ of $s$ is $e$. Elements of the form of

$$v = \left( \ldots, x_e, \ldots \right) = \left\{ \ldots, h_x, \ldots \right\}$$

are multiplications and all such elements form a normal subgroup,
the basis group $V(B, d) = V$ of $\Sigma$. The $h_e$ of $y$ are called the factors of $y$. Elements of the form of $s$ are permutations and all such elements form a subgroup, the permutation group, $S(B, C) = S$ of $\Sigma(H; B, d, C)$. Cycles of $s$ will also be written as $(x_1, \cdots, x_n)$ and $(\cdots, x_{-1}, x_0, x_1, \cdots)$. Baer [2] has shown that the normal subgroups of $S(B, C)$ are the alternating group, $A = A(B, d)$, and $S(B, D)$ where $d \leq D \leq C$. Let $E$ be the identity of $\Sigma$, $I$ the identity of $S$.

3. Normal subgroups of $\Sigma$ contained in the basis group.

**Theorem 1.** Let $G_1 \subseteq G$ be normal subgroups of $H$ such that $G/G_1$ belongs to the center of $H/G_1$. Let $N$ be the set of multiplications with a finite number of different from $e$ factors subject to the conditions that the factors are in $G$ and their product, in any order, belongs to $G_1$. $N$ is a normal subgroup of $\Sigma$ contained in the basis group. Conversely, let $N$ be a normal subgroup of $\Sigma$ contained in the basis group. For every $v$ of $N$ form all possible products of its nonidentity factors. This set $G_1$ is a normal subgroup of $H$. The set $G$ consisting of every element of $h$ that occurs as a factor in any $v$ of $N$ is also a normal subgroup of $H$. Further, $G/G_1$ belongs to the center of $H/G_1$.

**Proof.** It is clear that $N$ is a subgroup. Let $y_1 = v_1s_1$ be any element of $\Sigma$. Then $y_1yv^{-1} = v_1s_1v_2s_2^{-1}v_1^{-1} = v_1v_2v^{-1}$ where $v_2$ belongs to $N$. Since $G/G_1$ is in the center of $H/G_1$ the product of the factors of $v_1v_2v^{-1}$ is in $G_1$ and $N$ is normal in $\Sigma$.

If $N$ is normal in $\Sigma$ then an argument similar to that in [3, pp. 204—207] shows it has the structure described in the theorem.

4. Normal subgroups of $\Sigma$ not contained in the basis group. The problem of finding all normal subgroups of $\Sigma$ will be concluded by finding those normal subgroups not in $V$.

**Lemma 1.** Let $M$ be normal in $\Sigma$, $V \not\supseteq M$. Then $N = M \cap V$ is normal in $\Sigma$ and $G = H$, i.e., the factors in any fixed position run through $H$ and $H/G_1$ is Abelian.

**Proof.** Choose $y = vs \in M$ such that $s \neq I$. Let $v = \{k_1, k_2, \cdots\}$ be arbitrary in $V$. Then the multiplication $y^{-1}v^{-1}vy$ is in $M$. Let $s$ send $x_i$ into $x_{i*}$ with $e \neq i_e$. The factor occurring in the position occupied by $x_{i*}$ is $h_e^{-1}k_{e}^{-1}h_e k_{i*}$. Since $v$ is arbitrary and $i_e \neq e$ we may choose the factors $k_{e}^{-1}, k_{i*}$ in such a way that the factor above is arbitrary in $H$.

**Lemma 2.** Let $M$ be normal in $\Sigma$, $V \not\supseteq M$. Then $P = M \cap S$ is normal in $S$ and $P \neq E$.

**Proof.** Let $y$ be an element of $M$ and $y = vs$ where $s \neq I$. Since $y$
has only a finite number of different from e factors, \( M \) must contain a \( y' \) conjugate to \( y \) where the finite cycles of \( y \) have been written in normal form, [1, p. 20]. If \( y \) contains an infinite cycle then \( y' \) contains an infinite cycle in the form,

\[
(\cdots, x_{-1}, x_0, x_1, \cdots).
\]

If \( y' \) contains a finite cycle then [1, pp. 35–36] \( M \) contains a permutation. If \( y' \) contains only infinite cycles let \( s_1 = (2, 3) \). Then the commutator

\[
(y')^{-1} s_1 y' s_1^{-1}
\]

belongs to \( M \).

**Theorem 2.** If \( M = N \cap P \), where \( N \) is as in Lemma 1 and \( P \) is a normal subgroup of \( S \), then \( M \) is normal in \( S \).

**Proof.** Let \( y = v s \) be any element of \( M \). Let \( y x = v s_1 \) be any element of \( S \). Then \( y x y_1^{-1} = y_1 v v_1^{-1} v s_1 s_1^{-1} v_1^{-1} = v_1 v s_2 v_1^{-1} \) where \( v_1 \in N, s_2 \in P \) since \( N \) is normal in \( S \) and \( P \) is normal in \( S \). It is sufficient to show \( v_1 s_2 v_1^{-1} \) is in \( M \). A computation shows

\[
v_1 s_2 v_1^{-1} = \{h_1, \cdots, h_e, \cdots\} \begin{pmatrix} x_1, & \cdots, & x_e, & \cdots \end{pmatrix} \{h_1^{-1}, \cdots, h_e^{-1}, \cdots\}
\]

where all but a finite number of the factors of \( v_3 \) are \( e \). Since \( H / G_1 \) is Abelian the product of the factors of \( v_3 \) is in \( G_1 \). Therefore, \( v_3 \in N, s_2 \in P \) and \( M \) is normal in \( S \).

**Theorem 3.** If \( V \supset M \), \( M \) normal in \( S \), \( M \cap S = P = S(B, D) \), \( d \leq D \leq N \), then \( M = N \cup P \).

**Proof.** Assume there exists \( y \in M, y = v s, \) and \( s \in P \). This means \( s \) moves \( D \) or more of the elements of \( U \). Then \( y \) is conjugate to \( y' \) of \( M \) where \( y' \) in its cyclic normal form has \( d \)-cycles written in the form

\[
(\cdots, x_{-1}, x_0, x_1, \cdots).
\]

Construct \( s_1 \) as follows. For each \( n \)-cycle, \( n \geq 3 \), of \( y' \) of the form
let $s_1$ have the cycle $(x_1, x_2)$. For each pair of 2-cycles
\[
(x_1, x_2), \quad (x_3, x_4)
\]
of $y'$ let $s_1$ have the cycle $(x_1, x_3)(x_2)(x_4)$. If there is a 2-cycle
\[
(x_\alpha, x_\beta)
\]
of $y'$ left over let $s_1$ send $s_\alpha, s_\beta$ into themselves. For each $d$-cycle of $y'$
\[
(x_0, x_1, x_2, \ldots)
\]
let $s_1$ have the cycles
\[
(x_0)(x_n, x_{-n})
\]
for $n = 1, 2, \ldots$
Form the commutator $(y'^{-1} s_1 y' s_1^{-1}) = y_1$ which is in $M$. For each $n$-cycle, $n \geq 3$, of $y'$, $y_1$ contains $(x_1, x_2, x_3)$. For each pair of 2-cycles $y_1$ contains $(x_1, x_2, x_3)$. For each $d$-cycle of $y'$, $y_1$ will contain
\[
(\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, \ldots)
\]
This shows that $M$ contains a permutation which moves the same number of $x$'s as the $s$ of $y = vs$, contradicting $P = S(B, D)$.
If $y = vs \in M$ then since $s \in M$, $s^{-1}$ also belongs to $M$, and $ys^{-1} = v \in M$.

**Theorem 4.** Let $M \subseteq \Sigma$, $M$ normal in $\Sigma$, $M \cap S = A(B, d)$, $M \cap V = N$ and $M/N \cong A(B, d)$. Then $M = N \cup A(B, d)$.

**Proof.** Since $M \supseteq (N \cup A)$ and $N \cup A$ is normal in $\Sigma$ it follows that $N \cup A$ is normal in $M$. Thus $N \cup A/N$ is normal in $M/N$ which is simple since $M/N \cong A(B, d)$. Therefore, $N \cup A = M$.

**Theorem 5.** Let $M \subseteq \Sigma$, $M$ normal in $\Sigma$, $V \supseteq M$, $M \cap S = A$, $M \cap V = N$, and $M/N$ not $\cong A$. Then $M = N \cup A \cup L$, where $L$ is the cyclic subgroup generated by
\[ y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix} \]

with \(a^2 \in G_1\). If \(L_1\) is the cyclic subgroup generated by

\[ y_1 = \begin{pmatrix} x_1, & x_2 \\ x_2, & bx_1 \end{pmatrix}, \]

where \(b^2 \in G_1\), \(M_1 = N \cup A \cup L_1\) is \(M\) if, and only if, \(a\) and \(b\) are in the same coset of \(G_1\).

**Proof.** \(M\) contains an element \(y = vs\) with \(s \in S(B, C), C \geq d,\) and \(s \in A\). Otherwise, for every \(y = vs\) of \(M\), \(s \in A\). Then every element of \(V \cup M\) would be of the form \(y = v_1 vs = v_2 s\) and \((V \cup M)/V \cong A\). But \((V \cup M)/V \cong M/N\) not \(\cong A\). Now if \(y = vs\) with \(s \in S(B, C)\) with \(C > d\) the method used to prove Theorem 3 will lead to \(M \cap S \neq A\), a contradiction. So assume \(s \in S(B, d)\). The product of any 2 elements of \(M\) outside \(N \cup A\) is in \(N \cup A\) since the permutation component is finite and even. Let \(x_1, x_2\) be two elements \(s\) leaves fixed. The permutations \(s^{-1}(x_1, x_2)\) belong to \(A = P \subset M\). Therefore, \(y s^{-1}(x_1, x_2) = v ss^{-1}(x_1, x_2)\) belongs to \(M\). There is an element \(v_1\) in \(N\) such that \(v_1v(x_1, x_2)\) can be reduced to the form

\[ y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix} \]

because the factors of elements of \(N\) are unrestricted except that the product is in \(G_1\). This element squared is in \(N \subset M\) so \(a^2 \in G_1\).

**Theorem 6.** Let \(M = N \cup A \cup L\), where \(N\) is as in Lemma 1 and \(L\) is the cyclic group generated by

\[ y = \begin{pmatrix} x_1, & x_2 \\ x_2, & ax_1 \end{pmatrix} \]

with \(a^2 \in G_1\). Then \(M\) is normal in \(\Sigma\).

**Proof.** It is sufficient to show \(v_2 y (vs)^{-1}\) belongs to \(M\) for all \(vs\) of \(\Sigma\) because \(y^2\) belongs to \(N\). This may be reduced to showing \(v s^{-1}\) and \(v y v^{-1}\) belong to \(M\). Let \(s\) be arbitrary in \(S\) and

\[ s = \begin{pmatrix} x_i, \ldots, x_i, & \ldots \\ x_1, \ldots, x_2, & \ldots \end{pmatrix}. \]

Then

\[ sy s^{-1} = \begin{pmatrix} x_i, & x_i \\ x_i, & ax_i \end{pmatrix}. \]
But there exists an $s_1$ of $A$ such that

$$s_1 = \left( x_i, \ldots, x_j, \ldots \right).$$

But $s_1 y s_1^{-1}$ belongs to $(A \cup L \cup A) \subseteq M$. Let $v$ be arbitrary in $V$ and $v = \{h_1, \ldots, h_e, \ldots\}$. The commutator $y^{-1}vyv^{-1} = \{a^{-1}h_2ah_1^{-1}, h_1h_2^{-1}, e, \ldots\}$ belongs to $N \subseteq M$ if $a^{-1}h_2ah_1^{-1}h_1h_2^{-1}$, belongs to $G_1$. Since $H/G_1$ is Abelian the desired result follows.

**Theorem 7.** If $C = \mathfrak{S}_n$, where $n$ is finite, and if $H$ is finite then the number of normal subgroups of $\Sigma$ is finite.

**Proof.** This follows from the fact that there are only a finite number of choices of (1) normal subgroups of $S(B, C)$, (2) normal subgroups of $H$, (3) cosets of $G_1$.

**Bibliography**


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