DIFFERENTIAL EQUATIONS INVOLVING A PARAMETRIC FUNCTION$^1$

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1. Introduction. It is the purpose of this paper$^2$ to find conditions on the function $f(t, u)$ under which the differential system

$$\begin{cases}
\frac{dz}{dt} + f[t, y(t) + z(t)] = 0, & t \in I. \\
z(0) = 0
\end{cases}$$

has a solution $z(t)$ on the unit interval $I$ for almost all choices of the function $y$ in the space $C$. Here $C$ denotes the space of functions which are continuous on the interval $I$: $0 \leq t \leq 1$ and which vanish at $t = 0$; and "almost all" means all except a set of Wiener measure$^3$ zero. Under the transformation

$$z(t) = x(t) - y(t)$$

the system (1.1) goes into the equivalent nonlinear integral equation

$$y(t) = x(t) + \int_0^t f[s, x(s)]ds, \quad t \in I,$$  

so that we are seeking conditions on $f$ which make (1.3) have a solution $x \in C$ for almost every choice of $y$ in $C$.

The simplest conditions of this type which we have found, and which do not force (1.3) to have a solution for every $y \in C$, are given in the following theorem.

**Theorem 1.** Let $f(t, u)$ have continuous first partial derivations $f_t$ and $f_u$ in the strip $R: 0 \leq t \leq 1$, $-\infty < u < \infty$, and let $f$ satisfy the three order of growth conditions

$$\begin{align*}
(1.4) & \quad f(t, u) \text{ sgn } u \geq -A_1 \exp (Bu^2) \quad \text{ in } R, \\
(1.5) & \quad f_u(t, u) + 4g(t, u) \leq 2\alpha^2 u^2 + A_2 \quad \text{ in } R,
\end{align*}$$

$^1$ This research was supported by the United States Air Force, through the office of Scientific Research of the Air Research and Development Command, under contract No. AF 18(603)-30. Reproduction in whole or in part is permitted for any purpose of the United States Government.

$^2$ The author wishes to thank Mr. James Yeh for carefully checking the manuscript of this paper.

$^3$ See, for instance, [4].

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\[ g(1, u) \leq -\frac{1}{2} \alpha u^2 \cot \beta - A_3 \quad \text{for all real } u, \]

where

\[ g(t, u) = \int_0^u f(t, v) \, dv \quad \text{in } \mathbb{R}, \]

and \( A_1, A_2, A_3, B, \alpha, \beta \) are positive constants with \( \alpha < \beta < \pi \) and \( B < 1 \).

Then it follows that corresponding to almost every choice of \( y \in \mathbb{C} \), the system (1.1) has a solution \( z \in \mathbb{C} \), (and of course, this is the only solution defined on the interval \( I \)).

The fact that this can apply in cases where classical theorems (i.e., theorems giving conditions under which there are solutions for all choices of \( y \) in \( C \)) do not apply is shown by the counter-example

\[ f(t, u) = \frac{1}{2} u^{1/3} \sin (u^{4/3}) + \frac{1}{2} u^{5/3} \cos (u^{4/3}) \]

\[ = \frac{3}{8} \frac{d}{du} [u^{4/3} \sin u^{4/3}] \quad \text{in } \mathbb{R}. \]

It is easy to see that (1.8) satisfies the conditions of the theorem, and hence that there are solutions of (1.1) for almost all \( y \) in \( C \) when \( f \) is given by (1.8). On the other hand, we shall show in \( \S 3 \) that there exists at least one \( y \) in \( C \) for which (1.1) has no solution. Thus Theorem 1 cannot be contained in any classical theorem.

2. Proof of Theorem 1. Assume that the hypotheses of the theorem are satisfied, choose \( \gamma = B^{-1} - 1 \), and let

\[ \phi(t, u) = (t + \gamma)^{-1/2} \exp \left\{ u^2 (t + \gamma)^{-1} \right\} \quad \text{in } \mathbb{R}. \]

Let

\[ G(t, u, \lambda) = g(t, u) + \lambda \phi(t, u), \quad (t, u) \in \mathbb{R}, \lambda \geq 0, \]

so that

\[ G_u(t, u, \lambda) = f(t, u) + 2\lambda(t + \gamma)^{-1}u\phi(t, u), \quad (t, u) \in \mathbb{R}, \lambda \geq 0 \]

and

\[ G_{uu} + 4G_t = f_u + 4g_t, \quad (t, u) \in \mathbb{R}, \lambda \geq 0. \]

From (2.1), (2.0), (1.4) we have

\[ G_u \text{ sgn } u \geq A_1 \]
when $|u| \geq \lambda^{-1}B^{-3/2}A_1$ and $t \in I$ and $\lambda > 0$, and since for fixed positive $\lambda$, $G_u$ is bounded for $|u| \leq \lambda^{-1}B^{-3/2}A_1$, $t \in I$, it follows that there is a positive function of $\lambda$ alone, $A(\lambda)$, such that

\begin{equation}
G_u(t, u, \lambda) \operatorname{sgn} u \geq -A(\lambda), \quad (t, u) \in R, \lambda > 0. \tag{2.5}
\end{equation}

Now (2.5) implies that for fixed positive $\lambda$, the integral equation

\begin{equation}
y(t) = x(t) + \int_0^t G_u(s, x(s), \lambda)ds \tag{2.6}
\end{equation}

has a solution $x \in C$ for each $y \in C$. For the equation clearly has a solution on some interval to the right of zero, and if this interval does not include $t = 1$, it must be open at the right hand end and the solution must become unbounded in the neighborhood of this point. But by (2.5), this would imply that $y$ would vary by unbounded amounts as $x$ did so, contrary to the assumption that $y$ is continuous.

Since for fixed positive $\lambda$, (2.6) has a solution for each $y$ in $C$, it follows from Theorem 3 of [3], (using Footnote 9), that

\begin{equation}
\int_C \exp \left\{J(x, \lambda)\right\}d\mu x = 1, \quad \lambda > 0, \tag{2.7}
\end{equation}

where

\begin{equation}
J(x, \lambda) = \int_0^1 K(s, x(s), \lambda)ds - 2G(1, x(1), \lambda) \quad \text{for } x \in C, \lambda \geq 0, \tag{2.8}
\end{equation}

and

\begin{equation}
K(t, u, \lambda) = \frac{1}{2}G_{u,u} - G_u^2 + 2G_t \quad (t, u) \in R, \lambda \geq 0. \tag{2.9}
\end{equation}

But by (2.9), (2.2), (2.3),

\begin{equation}
\lim_{\lambda \to 0^+} K(t, u, \lambda) = K(t, u, 0) = \frac{1}{2}f_u + 2g_t - f^2 \quad \text{in } R, \tag{2.10}
\end{equation}

and for each fixed $x$ in $C$, we have from (2.8), (2.10), (2.1), (2.2), (2.3), by bounded convergence

\begin{equation}
\lim_{\lambda \to 0^+} J(x, \lambda) = J(x, 0) = \int_0^1 K(s, x(s), 0)ds - 2g(1, x(1)). \tag{2.11}
\end{equation}
Moreover, it follows from (2.8), (2.9), (2.3), (2.1), (1.5), (1.6), that
\[
J(x, \lambda) \leq \frac{1}{2} \int_0^1 \left\{ f_u(s, x(s)) + 4g_t(s, x(s)) \right\} ds - 2g(1, x(1))
\]
(2.12)
\[\leq \log Q(x) + \frac{1}{2} A_2 + 2A_3 \quad \text{for } x \in C, \lambda > 0\]
where for \(x \in C\),
\[(2.13) \quad Q(x) = \exp \left\{ \alpha^2 \int_0^1 [x(s)]^2 ds + \alpha [x(1)]^2 \cot \beta \right\}.
\]
Finally, we show that \(Q(x)\) is integrable over \(C\), by applying the transformation
\[y(t) = x(t) - \alpha \int_0^t \cot [\alpha s + \beta - \alpha] x(s) ds, \quad t \in I\]
to the Wiener integral of unity, using Theorem A of [2]. We obtain (using (2.13))
\[1 = \int_C 1 d\omega y = \exp \left\{ - \frac{1}{2} \alpha \int_0^1 \cot [\alpha s + \beta - \alpha] ds \right\} \int_C Q(x) d\omega x,
\]
so that the integrability of \(Q\) is established.

Now we take limits in (2.7) as \(\lambda \to 0^+\), using (2.11), (2.12) and dominated convergence, and thus establish that (2.7) holds even when \(\lambda = 0\). Hence it follows from Theorem 3 of [3], (using Footnote 9) and from (2.8), (2.9), (2.1), (2.2), (2.3), that the integral equation (1.3) has solutions \(x \in C\) for almost all \(y \in C\). But (1.3) is equivalent to (1.1) by the transformation (1.2), and the theorem is proved.

3. A counterexample. We shall now show that when \(f(t, u)\) is given by (1.8), there exists a function \(y \in C\) such that (1.3) (and hence also (1.1)), has no solution in \(C\). We begin by constructing a certain function \(x\) which does not belong to \(C\) because it becomes infinite as we approach \(t = 1\).

Let
\[(3.1) \quad t_n = 1 - n^{-1/3} \quad \text{and} \quad u_n = (2n\pi)^{3/4}, \quad n = 1, 2, 3, \ldots,\]
let \(M\) be the set of monotonically increasing functions defined on \([0, 1)\) which satisfy
\[(3.2) \quad x(0) = 0, \quad x(t_n) = u_n, \quad n = 1, 2, \ldots,\]
and let \(M_c\) be the subset of \(M\) consisting of those elements of \(M\) which are continuous on \([0, 1)\). Define the functionals \(Q_n(x)\) by
\( Q_n(x) = \int_{t_n}^{t_{n+1}} f[x(s)] ds, \quad x \in M, \)

where \( f(u) = f(t, u) \) is given by (1.8). We shall show the existence of an element \( x \) of \( M_c \) for which

\( Q_n(x) = u_n - u_{n+1} \)

for sufficiently large \( n \); i.e., for which

\( x(t_{n+1}) - x(t_n) + Q_n(x) = 0 \)

for sufficiently large \( n \).

To show that there is an element of \( M_c \) for which (3.4) holds for all sufficiently large \( n \), consider a particular interval \([t_n, t_{n+1}]\) and an element \( x_1 \) of \( M \) which is constant on \([t_n, t_{n+1})\), so that

\( x_1(t) = u_n, \quad t_n \leq t < t_{n+1}. \)

Then we have by (3.3), (3.1), (1.8),

\[ Q_n(x_1) = f(u_n)[t_{n+1} - t_n] = \frac{1}{2} u_n^b (t_{n+1} - t_n) > 0 > u_n - u_{n+1}. \]

Now it is clear that \( M_c \) is dense in \( M \) in the \( L_1[t_n, t_{n+1}] \) topology, and also that \( Q_n(x) \) is continuous in the \( L_1[t_n, t_{n+1}] \) topology applied to the space \( M \), since \( u_n \leq x(t) \leq u_{n+1} \) when \( x \in M \) and \( t \in [t_n, t_{n+1}] \). Hence there is an element \( x_2 \in M_c \) for which \( Q_n(x_2) \) differs as little as we please from \( Q_n(x_1) \), and in particular

\( Q_n(x_2) > u_n - u_{n+1}. \)

To obtain an \( x \) where the inequality goes the other way, we now set

\( x_3 = u'_n = [(2n + 1)\pi]^{3/4}, \quad t_n < t < t_{n+1}, \)

so that we have by (3.3), (3.1), (1.8),

\[ Q_n(x_3) = f(u'_n)[t_{n+1} - t_n] \]

\( = -\frac{1}{2} [(2n + 1)\pi]^{5/4} [n^{-1/3} - (n + 1)^{-1/3}], \)

\[ \leq -\frac{1}{6} [2n\pi]^{5/4} [n + 1]^{-4/3}, \]

while

\( u_n - u_{n+1} = (2n\pi)^{3/4} - [2(n + 1)\pi]^{3/4} \geq -\frac{3}{4} (2\pi)^{3/4} n^{-1/4}. \)
It is clear that there exists a positive integer \( N \) such that for \( n > N \), the last member of (3.8) is greater than the last member of (3.7), so that we have

\[
Q_n(x_3) \leq u_n - u_{n+1} \quad \text{if } n > N.
\]

Hence it follows from the continuity of \( Q_n \) in the \( L_1[t_n, t_{n+1}] \) topology and the density of \( M_c \) in \( M \) that there exists an element \( x_4 \in M_c \) for which

\[
(3.9) \quad Q_n(x_4) \leq u_n - u_{n+1} \quad \text{if } n > N.
\]

Now if we put

\[
x(t) = \lambda x_2(t) + (1 - \lambda)x_4(t), \quad t \in I,
\]

it follows from (3.6) and (3.9) and the continuity of \( Q_n(x) \) that there is a value of \( \lambda \) on \((0, 1)\) for which (3.4) holds, if \( n > N \). Thus for \( n > N \), it is possible to choose \( x(t) \) on each interval \([t_n, t_{n+1}]\) so as to make (3.4) hold, and these choices can be made independently for each \( n > N \). Let such choices be made for each interval \([t_n, t_{n+1}]\), \( n > N \), and choose \( x \) on the previous intervals in any way which makes \( x \in M_c \).

Using this choice of \( x \), we now define \( y \) on \([0, 1)\) by substituting this \( x \) in (1.3). It now follows from (3.4) that for \( n > N \), (3.5) holds, and from (3.3), (3.5), (1.3) that \( y(t_n) \) is independent of \( n \) for \( n > N \). Setting \( y(1) \) equal to this constant value, we have

\[
(3.10) \quad y(t_n) = y(1), \quad n > N,
\]

and \( y(t) \) is now defined everywhere on \( I \).

To show that \( y \) is left continuous at \( t = 1 \), we assume \( n > N \) and \( t_n \leq t \leq t_{n+1} \), and write, using (3.10), (1.8), (3.1) and the monotonicity of \( x \),

\[
|y(t) - y(1)| = |y(t) - y(t_n)| \\
\leq x(t_{n+1}) - x(t_n) + \int_{t_n}^{t_{n+1}} |f(x(s))| \, ds \\
\leq u_{n+1} - u_n + \int_{t_n}^{t_{n+1}} \frac{5}{3} \frac{u_{n+1}}{u_{n+1}} ds \\
= [(2n + 1)\pi]^{3/4} - (2n\pi)^{3/4} + [2(n + 1)\pi]^{5/4}[n^{-1/3} - (n + 1)^{-1/3}] \\
\to 0 \quad \text{as } n \to \infty,
\]

and it follows that \( y \in C \). Moreover, this choice of \( y \) corresponds to no solution \( x \in C \) of (1.3), since the solution is the chosen \( x \) and is unique on each interval \([0, 1 - \varepsilon]\), and any solution on \([0, 1]\) would have to
agree on $[0, 1)$ with this unbounded $x$ that we chose. Thus it is established that if $f$ is given by (1.8), the equation (1.3) does not have a solution for every $y$ in $C$.

4. Conclusion. For simplicity, we did not state Theorem 1 in its most general form, and it is easy to see from the proof of the theorem that we can weaken two of the hypotheses a little.

Theorem 2. If we weaken conditions (1.4) and (1.5) of Theorem 1 by replacing them by conditions

\begin{align*}
(1.4') \quad f(t, u) \, \text{sgn } u &\geq -A_1 \exp \left(\frac{u^2}{t + \gamma} \right) \quad \text{in } R, \\
(1.5') \quad f_u(t, u) + 4g(t, u) &\leq 2a^2u^2 + A_2 + \left\{ \max \left[0, f(t, u)\text{sgn } u\right] \right\}^2 \text{in } R,
\end{align*}

where $\gamma$ is a positive constant, it follows that the conclusion of Theorem 1 holds.

Whether or not Theorem 2 is really more general than Theorem 1 is still an open question, as the author has not yet found any function which satisfies the hypotheses of Theorem 2 but not those of Theorem 1.

In a previous paper, [1], the author raised the question whether

$$y(t) = x(t) + \int_0^t [x(s)]^2 ds$$

has solutions for almost every $y$ in $C$, and he pointed out two other questions which are equivalent to this one. These questions are not answered by this paper, since $f(t, u) = u^2$ does not satisfy condition (1.6). They have, however, recently been answered in the negative in an unpublished paper by D. A. Woodward.

References


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