QUASI-REFLEXIVE SPACES[1]

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1. **Introduction.** For a Banach space X, let π denote the canonical isomorphism of X into X^{**}. An example was given by James [5] with $X^{**}/\pi(X)$ finite dimensional. The example has the further property that X and X^{**} are isomorphic. We consider the class of spaces which we call quasi-reflexive spaces of order n in which $X^{**}/\pi(X)$ is (finite) n-dimensional. We characterize such spaces by the compactness in a suitable weak topology of the unit ball under an equivalent norm. Quasi-reflexive spaces are mth conjugate spaces for all integers m.

We show that if E is a closed linear manifold in a space X, then X is quasi-reflexive of order n if and only if E and X/E are quasi-reflexive of orders m and p respectively and $n=m+p$. The case $n=0$ would then be the corresponding result for reflexive spaces [6]. It is further shown that the additivity of dimension holds in a natural interpretation even when $X^{**}/\pi(X)$ is infinite dimensional.

Any weakly complete quasi-reflexive space is reflexive. Every nonseparable quasi-reflesive space has a nonseparable reflexive subspace.

2. **Notation.** Let X by a Banach space. Let π be the canonical isomorphism of X into X^{**}, its second conjugate space. We denote by $H(X)$ the quotient space $X^{**}/\pi(X)$. For a subset A of X, A^+ will designate the annihilator of A in X^*, and A^{++} the annihilator of A^+ in X^{**}. For a set B' in X^*, B'^- will denote the annihilator of B' in X. If T is a linear mapping from X into a Banach space Y, T^* will denote the adjoint mapping of Y^* into X^*. We use the notation $X \cong Y$ to indicate the isomorphism of two Banach spaces X and Y in the sense of [1].

In conjunction with any Banach space X we associate a cardinal number Ord (X) which is the cardinal number of a Hamel basis for $H(X)$. If Ord (X) is a finite number n, we say that X is *quasi-reflexive* of order n.

If A and B are closed subspaces of X, we denote the linear space sum of A and B by $A+B$. We use $A \oplus B$ for the linear space direct sum.

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3. **Quasi-reflexive and second conjugate spaces.** We first utilize various results of Dixmier [2] to characterize spaces isomorphic to a second conjugate space, in a manner similar to that used by him to characterize spaces isomorphic to first conjugate spaces.

Let Q' be a linear manifold in X^*. By $\sigma(X, Q')$ we mean the least fine topology in which all the $x^* \in Q'$ are continuous.

3.1. THEOREM. *The following statements are equivalent:*

(1) *X is isomorphic to a second conjugate space.*

(2) *There is an equivalent norm for X such that $X^* = Q' \oplus R'$, where Q' is a total closed linear manifold such that the unit ball of X is compact in $\sigma(X, Q')$, and R' is a regularly closed linear manifold.*

Suppose that $X \cong Y^{**}$. Let π and π_1 be the canonical isomorphisms of Y and Y^* into Y^{**} and Y^{***}, respectively. Then, as noted by Dixmier [2, Theorem 15], $Y^{***} = \pi_1(Y^*) \oplus \pi(Y)^+$. Now $\pi(Y)^+$ is regularly closed and $\pi_1(Y^*)$ is total over Y^{**}. Give X the norm induced by the isomorphism, and identify X and Y^{**}. The unit ball of Y^{**} is w^*-compact by Alaoglu's theorem, hence is compact in the topology $\sigma(Y^{**}, \pi_1(Y^*))$.

Assume (2), and consider X in the norm given by (2). Let $E = R'^-$. Let α be the restriction of $x^* \in X^*$ to the domain of definition E. The kernel of α is $E^+ = R'$. Let $z^* \in E^*$. Extend z^* to $x^* \in X^*$, and write $x^* = x_1^* + x_2^*$ with $x_1^* \in Q'$ and $x_2^* \in R'$. Then $z^* = \alpha(x^*) = \alpha(x_1^*)$. Thus α is a 1-1 bi-continuous mapping of Q' onto E^*. By Theorems 14 and 16 of [2], X is equivalent to Q'^*, and X is isomorphic to E^{**}.

From Theorem 3.1 the following is readily deduced.

3.2. COROLLARY. *X is isomorphic to X^{**} and not reflexive if and only if $X^* = Q' \oplus R'$ as in Theorem 3.1 where $R' \neq (0)$ and Q' is isomorphic to X^*.*

The example of James [5] provides such an X.

3.3. THEOREM. *The following statements are equivalent:*

(1) *X is quasi-reflexive of order n.*

(2) *There is an equivalent norm for X such that $X^* = Q' \oplus R'$; where Q' is a total closed linear manifold such that the unit ball of X is compact in $\sigma(X, Q')$, and R' is an n-dimensional linear manifold.*

Assume (1). Then $X^{**} = \pi(X) \oplus L''$, where L'' is n-dimensional. Let $Q' = L''^-$. It is readily verified that Q' is total. Let $x_1^{**}, \cdots, x_n^{**}$ be a basis for L'' and select $x_1^*, \cdots, x_n^*$ in X^* such that $x_i^{**}(x_j^*) = \delta_{ij}$, $i, j, \cdots, n$. Let R' be the subspace of X^* generated by $x_1^*, \cdots, x_n^*$. It is easily seen that $X^* = Q' \oplus R'$. By Theorems 10, 6, 8″ and 14 of [2], (2) follows.

Assume (2). By Theorems 14 and 11 of [2], $X^{**}=\pi(X)\oplus Q'^{+}$. Let $x_1^*, \cdots, x_n^*$ be a basis for R'. Select $x_1^{**}, \cdots, x_n^{**}\in Q'^{+}$ such that $x_i^{**}(x_j^*)=\delta_{ij}$, $i, j=1, \cdots, n$. Let L'' be the subspace of X^{**} generated by $x_1^{**}, \cdots, x_n^{**}$. If $x^{**}\in Q'^{+}$, $x^{**}-\sum_{j=1}^{n} x^{**}(x_j^*)x_j^{**}=0$ so that $Q'^{+}=L''$. Thus X is semi-reflexive of order n.

3.4. LEMMA. *X is quasi-reflexive of order n if and only if X^* is quasi-reflexive of order n.*

By [2, Theorem 15], $H(X^*)\cong\pi(X)^{+}$. But $\pi(X)^{+}\cong[H(X)]^*$. Thus $H(X)$ has dimension n (finite) if and only if $H(X^*)$ has dimension n.

3.5. THEOREM. *Let X be quasi-reflexive of order n. Then there exist Banach spaces $X^{(j)}$, $j=0, \pm 1, \pm 2, \cdots$, such that $X^{(0)}=X$ and $X^{(j+1)}=(X^{(j)})^*$. Further more each $X^{(j)}$ is quasi-reflexive of order n.*

Let X be quasi-reflexive of order n. By Lemma 3.4, $X^*, X^{**}, \cdots,$ are quasi-reflexive of order n. By Theorems 3.1 and 3.3, X is the conjugate space of a Banach space $X^{(-1)}$. By Lemma 3.4, $X^{(-1)}$ is quasi-reflexive of order n. This procedure can be continued.

Let $X_1^*\cong X_2^*$ where X_1 and X_2 are Banach spaces. It appears to be an open question [1, p. 243] whether or not this implies $X_1\cong X_2$.

3.6. THEOREM. *Let X_1 and X_2 be Banach spaces where X_1 is quasi-reflexive. If X_1^* is isomorphic to X_2^*, then X_1 is isomorphic to X_2.*

Let X_1 be quasi-reflexive of order m. By Lemma 3.4, X_1^* is quasi-reflexive of order m. Then so is X_2^* since $X_1^*\cong X_2^*$. By Lemma 3.4, X_1 and X_2 are then quasi-reflexive of order m. We write $X_i^{**}=\pi(X_i)\oplus G_i''$, $i=1, 2$, where G_1'' and G_2'' are m-dimensional with bases $x_1^{**}, \cdots, x_m^{**}$, $y_1^{**}, \cdots, y_m^{**}$, respectively. The given isomorphism induces an isomorphism U of X_1^{**} onto X_2^{**}. There exists [7, p. 161] an isomorphism V of X_2^{**} onto Y_2^{**} such that $VU(x_i^{**})=y_i^{**}$, $i=1, \cdots, m$. Let α be the canonical homomorphism of X_2^{**} onto X_2^{**}/G_2''. Then the mapping αVU induces an isomorphism of X_1^{**}/G_1'' onto X_2^{**}/G_2''. Then $\pi(X_1)$ is isomorphic to $\pi(X_2)$.

4. Subspaces and quotient spaces.

4.1. THEOREM. *Let E be a closed linear manifold in X. Then $\pi(X)+E^{++}$ is a closed linear manifold in X^{**}. Also*

$$H(E)\cong(\pi(X)+E^{++})/\pi(X), \tag{1}$$

$$H(X/E)\cong X^{**}/(\pi(X)+E^{++}), \tag{2}$$

where π is the canonical isomorphism of X into X^{**}.

Let μ be the canonical isomorphism of X/E into $(X/E)^{**}$. Let β be the canonical homomorphism of X onto X/E. We show first that

$$\mu\beta = \beta^{**}\pi. \tag{3}$$

For let $x \in X$, $w^* \in (X/E)^*$. Then $\beta^{**}\pi(x)(w^*) = \beta^*(w^*)(x) = w^*\beta(x) = \mu\beta(x)(w^*)$. From this we obtain

$$\beta^{**-1}\mu(X/E) = \pi(X) + E^{++}. \tag{4}$$

For note that $\beta(X) = X/E$, $\mu\beta(X) = \mu(X/E)$. As the kernel of β^{**} is E^{++}, we derive (4) from (3). This shows that $\pi(X) + E^{++}$ is closed in X^{**}.

Let R be the mapping of X^* onto E^* defined by the rule that $R(x^*)$ is the restriction of $x^* \in X^*$ to the domain of definition E. Then R^* is an isomorphism of E^{**} into X^{**} with range E^{++}. Let σ be the canonical isomorphism of E into E^{**}. We show

$$R^*\sigma(x) = \pi(x), \qquad x \in E. \tag{5}$$

For let $x^* \in X^*$. $R^*\sigma(x)(x^*) = R(x^*)(x) = x^*(x) = \pi(x)(x^*)$.

Let α be the canonical homomorphism of X^{**} onto $H(X)$. Note that $\pi(X) \cap E^{++} = \pi(E)$. Thus the kernel of αR is $R^{*-1}\pi(E) = \sigma(E)$ by (5). The range of αR is $\alpha(E^{++})$. Thus αR defines an isomorphism of $H(E)$ onto $(\pi(X) + E^{++})/\pi(X)$. Clearly the isomorphism is bi-continuous. This shows (1).

Let γ be the canonical homomorphism of $(X/E)^{**}$ onto $H(X/E)$. The mapping $\gamma\beta^{**}$ takes X^{**} onto $H(X/E)$. Its kernel is $\{x^{**} \in X^{**} \mid \beta^{**}(x^{**}) \in \mu(X/E)\} = \pi(X) + E^{++}$ by (4). This provides a bi-continuous isomorphism of $X^{**}/(\pi(X) + E^{++})$ onto $H(X/E)$.

4.2. COROLLARY. Ord $(X) =$ Ord $(E) +$ Ord (X/E). *Thus X is quasi-reflexive if and only if E and X/E are quasi-reflexive.*

This follows immediately. The well-known result that X is reflexive if and only if E and X/E are reflexive [6] is also an immediate consequence.

4.3. COROLLARY. *E is reflexive if and only if $E^{++} \subset \pi(X)$. X/E is reflexive if and only if $\pi(X) + E^{++} = X^{**}$. If X/E is reflexive then $H(E) \cong H(X)$. If E is reflexive then $H(X/E) \cong H(X)$.*

4.4. THEOREM. *A quasi-reflexive space is reflexive if and only if it is weakly complete.*

Let X be quasi-reflexive of order n and weakly complete. It is sufficient to show [3] that the unit ball of X is sequentially weakly compact. Let $x_i \in X$, $\|x_i\| \leq 1$, $i = 1, 2, \cdots$, and let X_1 be the closed linear manifold generated by $\{x_i\}$. By Corollary 4.2, X_1 is quasi-reflexive of order $m \leqq n$. Since X_1 is a closed linear manifold in a weakly complete space, it is weakly complete. Moreover X_1^{**} is separable, since $X_1^{**} = \pi(X_1) \oplus F''$ with F'' m-dimensional. Thus X_1^* is separable and thus by [4] X_1 is reflexive. Thus there exists a subsequence $\{y_i\}$ of $\{x_i\}$ and an $x_0 \in X_1$ such that $y^*(y_i) \rightarrow y^*(x_0)$ for $y^* \in X_1^*$. By restriction we then have $x^*(y_i) \rightarrow x^*(x_0)$ for $x^* \in X^*$.

4.5. Corollary. *A quasi-reflexive space X with an unconditional basis is reflexive.*

By the argument in Theorem 4.4, X^{**} is separable. Hence Corollary 1 of [5] asserts that X is reflexive.

4.6. Theorem. *If X is a nonseparable space which is quasi-reflexive, then X has a nonseparable reflexive subspace Z such that X/Z is separable.*

Suppose X is quasi-reflexive of order $n \geqq 1$. By Theorem 3.5 $X \cong Y^*$ where Y is quasi-reflexive of order n. As seen in the proof of Theorem 4.4, Y is not separable. Let $\mathfrak{P}$ be the collection of all separable closed linear manifolds in Y. We claim that not all elements of $\mathfrak{P}$ are reflexive. For otherwise it follows that the unit ball of Y is weakly compact and hence that Y is reflexive [3]. Alternatively one can show that Y is weakly complete and hence, by Theorem 4.4, reflexive. Let $Y_1 \in \mathfrak{P}$, Y_1 not reflexive. By Corollary 4.2, Y/Y_1 is quasi-reflexive of order $n_1 < n$. Now Y/Y_1 is not separable, since Y is not separable and Y_1 is separable. Also $(Y/Y_1)^* \cong Y_1^+$, so Y_1^+ is a nonseparable subspace of X which is quasi-reflexive of order n_1 by Lemma 3.4.

By [6, p. 573], $X/Y_1^+ \cong (Y_1^{+-})^* \cong Y_1^*$. As seen in the proof of Theorem 4.4, Y_1^* is separable. Set $X_1 = Y_1^+$. If X_1 is not reflexive there exists X_2 a closed linear manifold in X_1 which is not separable, X_1/X_2 separable and X_2 quasi-reflexive of order $n_2 < n_1$. We continue this procedure in this way until we obtain X_m a reflexive, nonseparable closed linear manifold of X_{m-1} where X_{m-1}/X_m is separable. Since $X/X_1 \cong (X/X_2)/(X_1/X_2)$ it follows that X/X_2 is separable. By continuing this argument we obtain X/X_m separable.

4.7. Corollary. *Let X be nonseparable and quasi-reflexive of order n. Then there exists a separable space Y, quasi-reflexive of order n, and a bounded linear mapping of X onto Y.*

By Theorem 4.6 and Corollary 4.2, in the notation of Theorem 4.6, we can take $Y = X/Z$.

5. A mapping theorem.

5.1. THEOREM. *Let X be a quasi-reflexive space. Then any bounded linear mapping T of X into l_1 is completely continuous.*

Let $\{x_n\}$ be a sequence in the unit ball of X and let X_1 be the closed linear manifold generated by the sequence $\{x_n\}$. It is sufficient to show that T is completely continuous on X_1. As in the proof of Theorem 4.4, X_1^* is separable. By Theorems 4.10 and 1.8 of [8] it is seen that any bounded linear mapping of a space X_1 with a separable conjugate space into l_1 is completely continuous on X_1.

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