AN EXTREMAL PROBLEM CONCERNING THE CENTRE OF GRAVITY OF A CONVEX DISC

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1. **Introduction.** The following question is answered in this paper. How far apart at most can the centroid of a convex disc be from one of its chords of given length provided the disc is restricted by a bound on its diameter?

The existence of an extremal domain is shown by proving in §2 the continuity of the functional in question and the compactness of the space of competing domains in §3. By suitable variations this maximizing domain is determined in §§4–6, and finally the sharp numerical bounds are computed in §§7 and 8, in particular for the distance of the centroid from a diameter.

2. **Continuity of the functional** y_R. We denote by R_p the exterior parallel region to a given bounded convex planar region R at distance p. For the corresponding areas Steiner's relation holds

$$A_p = A + Lp + \pi p^2,$$

where A and L are area and perimeter of R. The space of all convex discs is metrized in the usual way by introducing the distance $\rho(R, R')$ between two of these domains as the minimum of all numbers p such that simultaneously $R \subset R_p'$ and $R' \subset R_p$. We show that in this topology the distance y_R of the centroid of a region R from a fixed straight line x in the plane of R is a continuous functional of the argument R provided that the area A of R does not vanish.

For this purpose we first enclose R in a closed square S of side s large enough such that $R_1 \subset S$. We denote by M the maximum of the distances of all points of S from the line x. Given $\epsilon > 0$ we can now determine $\delta(\epsilon) > 0$ satisfying the following conditions.

(a) $$R_{2\delta} \subset S,$$

(b) $$4s\delta + \pi\delta^2 < \operatorname{Min}\left\{\frac{1}{2}A, \frac{\epsilon A}{12M}, \frac{A}{2}\left(\frac{\epsilon}{3M}\right)^{1/2}\right\}.$$

Let now R' denote an arbitrary convex region with area A' and perimeter L' such that $\rho(R, R') < \delta$. We have then by assumption (a)

$$R' \subset R_\delta' \subset R_{2\delta} \subset S$$

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and therefore due to the convexity

$$L < 4s \quad \text{and} \quad L' < 4s.$$

Denote by (I) the intersection of R and R' with area I and let

$$A = I + \alpha \quad \text{and} \quad A' = I + \alpha'.$$

Since for the corresponding domains the inclusions

$$(\alpha) \subset R'_\delta - R' \quad \text{and} \quad (\alpha') \subset R_\delta - R \quad \text{hold,}$$

we have

$$\alpha \leqq L'\delta + \pi\delta^2 < 4s\delta + \pi\delta^2, \tag{1}$$

and

$$\alpha' \leqq L\delta + \pi\delta^2 < 4s\delta + \pi\delta^2. \tag{2}$$

Furthermore the following estimates are needed:

$$\left| \int_{(I)} y dA \right| < MI \leqq MA, \tag{3}$$

$$\left| \int_{(\alpha)} y dA \right| < M\alpha < M(4s\delta + \pi\delta^2), \tag{4}$$

$$\left| \int_{(\alpha')} y dA \right| < M\alpha' < M(4s\delta + \pi\delta^2). \tag{5}$$

From $R \subset R'_\delta$ and assumption (b) we conclude that

$$A \leqq A' + \delta L' + \pi\delta^2 \leqq A' + 4s\delta + \pi\delta^2 < A' + A/2,$$

and therefore by subtracting $A/2$,

$$A' > A/2. \tag{6}$$

Substituting

$$y_R = \frac{1}{A}\int_R y dA \quad \text{and} \quad y_{R'} = \frac{1}{A'}\int_{R'} y dA$$

we obtain

$$|y_R - y_{R'}|$$
$$= \frac{1}{AA'}\left| \alpha' \int_{(I)} + I \int_{(\alpha)} + \alpha' \int_{(\alpha)} - \alpha \int_{(I)} - I \int_{(\alpha')} - \alpha \int_{(\alpha')} \right| \cdot$$

Replacing on the right every term by its absolute value and making

use of the above inequalities $(1-6)$ and of the assumption (b) it is easily seen that each of the six terms is less than $\epsilon/6$. Consequently $|y_R - y_{R'}| < \epsilon$ for every R' satisfying $\rho(R, R') < \delta$, which proves the continuity.

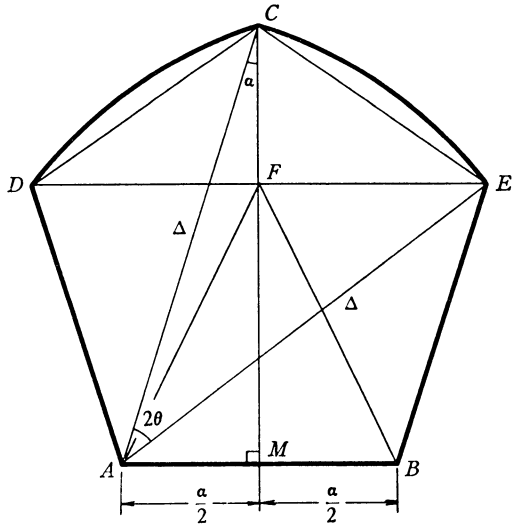
The requirement $A \neq 0$ is necessary as seen by the example of a narrow triangle and a thin rod arbitrarily close to each other but with centroids apart.

3. **Existence of an extremal domain.** Denote by (C) the class of all convex planar domains with diameter less than or equal to a given bound Δ and having the chord AB of length $\overline{AB} = a$ $(a \leqq \Delta)$. (C) is the set of competing domains from which we want to select a region the centroid of which has maximum distance from AB. Since (C) is obviously bounded (i.e. all regions can be enclosed in a large square) the set of their distances y from the line AB is bounded and possesses a smallest upper bound Y. According to Blaschke's selection principle there exists a converging sequence of regions belonging to (C), $R(n) \to R$, such that $\lim y_n = Y$. From the continuity of y we conclude that Y is the height above AB of the centroid of R. Finally R belongs itself to the class (C) of competing regions: Let D_n and D be the diameters of $R(n)$ and R respectively. The inequalities $D_n \leqq \Delta$ $(n = 1, 2, \cdots)$ carry over to the limit region, $D \leqq \Delta$, because of the known continuity of the diameter, and since AB is a chord of every $R(n)$ it is also a chord of R. Thus the space (C) is compact, which assures the existence of a solution to our extremal problem.

4. **Description of the classes (P) and (Q).** In order to determine this solution we first select from the class (C) a certain one-parameter subset (P) (see figure). Denote by M the midpoint of AB and by C the point of intersection of the semicircles about A and B on one side of AB with radius Δ. An arbitrary parallel line to AB between a and C intersects these semicircles in 4 points. Let the two symmetrical ones closest to the line CM be denoted by D and E, where D is on the same side of CM as A and E on the side of B. The resulting pentagon $ABECD$, where AB, BE, and AD are straight segments and CD and CE are arcs on our semicircles, is said to belong to class (P) provided that

$$L = \overline{DE} < \Delta. \tag{7}$$

Then its diameter is Δ and hence $(P) \subset (C)$; for if X denotes an arbitrary point on the segment BE and Y any other point in our plane then $\overline{XY} \leqq \operatorname{Max}(\overline{YB}, \overline{YE})$, and if Y is situated on the arc CD then $\overline{YB} = \Delta$ and $\overline{YE} \leqq \overline{DE} < \Delta$, thus $\overline{XY} \leqq \Delta$.



Let α denote the angle ACM $(0 \leqq \alpha \leqq \pi/6)$ and 2θ the angle CAE. Condition (7) is expressed by the following inequalities:

$$0 \leqq \theta < \frac{1}{2} \arcsin\left(\frac{1}{2} + \sin\alpha\right) - \frac{\alpha}{2} \equiv \theta_u(\alpha). \tag{8}$$

The angle θ is a parameter specifying a certain member $P(a, \Delta, \theta)$ of class (P).

The class of all pentagons $ABECD$ with arbitrary θ,

$$\left(0 \leqq \theta \leqq \frac{1}{2}\left(\frac{\pi}{2} - \alpha\right)\right) \text{ will be denoted by } (Q),\ (P) \subset (Q). \tag{9}$$

Let F be the intersection of MC and DE, and let $MF = h$, $MC = H$. We consider in the next section the following triangular regions and their areas

$$\begin{aligned} ABF &= A_1 = \frac{1}{2}\, ah, \qquad AFD + BFE = A_2, \\ DEC &= A_3 = \frac{1}{2}\, L(H - h). \end{aligned} \tag{10}$$

The heights of their respective centroids above AB are

$$y_1 = \frac{1}{3}\, h, \qquad y_2 = \frac{2}{3}\, h, \qquad y_3 = \frac{2}{3}\, h + \frac{1}{3}\, H. \tag{11}$$

We also introduce the ratios $a/\Delta=\lambda$ and $y_Q/\Delta=\eta$, where y_Q is the distance of the centroid from the chord AB of a domain of class (Q). It is easily verified that

$$\sin\alpha = \frac{a}{2\Delta} = \frac{1}{2}\lambda, \tag{12}$$

and

$$\eta(\alpha,\theta)=\frac{1}{3}\,\frac{4\sin\theta\cos(\alpha+\theta)+2\sin\alpha\cos^2(\alpha+2\theta)-\sin\alpha\cos^2\alpha}{2\theta+2\sin\alpha\cos(\alpha+2\theta)-\sin\alpha\cos\alpha}. \tag{13}$$

5. Lemma. *The maximum among the distances from AB of the centroids of the regions of class (Q) is attained by a domain of class (P).*

Proof. The function (13) is too difficult to handle directly; we consequently replace a region R of (Q) by the *straight* pentagon $ABECD$ as follows.

Consider first the function

$$f(x) = +\left[(1-x^2)\left(\frac{3}{4}-x-x^2\right)\right]^{1/2},$$

which is monotonically decreasing in the interval

$$I: 0 \leqq x \leqq 1/2.$$

The tangent $t(x)=-3^{1/2}x/3+3^{1/2}/2$ at $(0, 3^{1/2}/2)$ does not intersect the curve again in this interval I. Hence

$$f(x) \leqq t(x) \text{ for } x \in I. \tag{14}$$

On the other hand the cubic polynomial

$$g(x) = 2x^3 + x^2 - \frac{3}{2}x + 1$$

is at $x=0$ above $t(x)$ and the graphs of $g(x)$ and $t(x)$ do not intersect in I. Therefore

$$g(x) > t(x) \text{ for } x \in I. \tag{15}$$

Combining (14) and (15) we find

$$2x^3 + x^2 - \frac{3}{2}x + 1 > +\left[(1-x^2)\left(\frac{3}{4}-x-x^2\right)\right]^{1/2} \quad \text{for } x \in I.$$

We divide this inequality by $3/4-x-x^2$ (>0 for $0\leqq x<1/2$), then add the expression $2x-((1-x^2)/(3/4-x-x^2))^{1/2}$ on both sides. Letting $x=\sin\alpha$ and

$$\rho(\alpha) = \frac{\cos\alpha}{(3/4 - \sin\alpha - \sin^2\alpha)^{1/2}}, \tag{16}$$

the result may be written in the following form:

$$\rho(\alpha)[\rho(\alpha) - 1] > 2\sin\alpha \text{ for } 0 \leqq \alpha < \pi/6. \tag{17}$$

Obviously $\rho(\alpha) > 0$ and $2\sin\alpha \geqq 0$ for $0 \leqq \alpha < \pi/6$; since the function $z(z-1)$ is negative for $0 < z < 1$, (17) implies therefore

$$\rho(\alpha) > 1 \text{ for } 0 \leqq \alpha < \pi/6. \tag{18}$$

Let now R denote a region of class (Q) not belonging to (P), i.e.

$$L \geqq \Delta. \tag{19}$$

In class $(Q)-(P)$ the maximum of h is obtained by the region with $L = \Delta$; hence it is easy to see that

$$h_{\max} = \Delta(3/4 - \sin\alpha - \sin^2\alpha)^{1/2}.$$

Combining this with $H = \Delta\cos\alpha$, (16), and (18), we obtain

$$H/h \geqq \rho(\alpha) > 1. \tag{20}$$

From (12) and (19) we find

$$a/L \leqq 2\sin\alpha. \tag{21}$$

Since the quadratic function $z(z-1)$ is increasing for $z > 1$ we gather from (20), (17), and (21) that

$$\frac{H}{h}\left(\frac{H}{h} - 1\right) \geqq \rho(\alpha)[\rho(\alpha) - 1] > 2\sin\alpha \geqq \frac{a}{L}\cdot$$

Multiplying by $h^2L/6$ and using (10) we find $A_3H/3 > A_1h/3$, or by (11): $A_3(y_3 - y_2) > A_1(y_2 - y_1)$. Adding $A_1y_1 + A_2y_2 + A_3y_3$ on both sides and dividing by $A_1 + A_2 + A_3$ we finally obtain

$$y_s > \frac{2}{3}h, \tag{22}$$

where y_s is the height above AB of the centroid of the pentagon $ABECD$ with straight sides. This has been proved for $L \geqq \Delta$ and $0 \leqq a < \Delta$, but is easily seen to hold also for $a = \Delta$.

We distinguish now two cases (the question whether both of them actually occur is immaterial here).

(i) Let the centroid of R be at a height greater than h. By letting the angle θ slightly decrease we then cut off from R two nearly triangular areas whose common centroid is arbitrarily close to the point

on MC at height $2h/3$. Since this is below F the centroid of the new region will be higher than before.

(ii) Let the centroid of R be on the line DE or below it at a height $y_R \leqq h$. Then we have $y_s < y_R$, since the segmental areas above DC and EC are added to the straight pentagon in order to obtain R. Hence taking into account (22) we find also $y_R > 2h/3$. Consequently by the same procedure of letting θ decrease as in (i) we again can raise the centroid of R.

Thus every region in $(Q)-(P)$ can be replaced by a region belonging to (Q) with higher centroid. The maximum of the continuous function (13) in the closed interval (9) will therefore occur in the subinterval (8), which proves our lemma. The value $\theta=0$ in (8) can also be excluded since a slight increase in θ obviously increases y_R.

6. **Variations within class** (C). We show next how a domain R of class (C) not belonging to class (P) can be transformed by certain variations (a)–(e) into another domain $R' \in (P)$ with an equal or greater y. From this we conclude that the extremal domain belongs to class (P) and is consequently identical to a region $P_{\max}$ for which (13) as a function of θ attains its maximal value.

(a) If R is not symmetrical with respect to the midperpendicular of AB we symmetrize it at this line. It is known [1] that in this process the diameter does not increase while y remains obviously unchanged.

(b) If the centroid G of the given region R lies on the chord $AB=a$ we replace R e.g. by a triangle with base a.

(c) If a is not a part of the boundary of R we remove that part of R lying on the opposite side of G. Thus we obtain a new domain R' also belonging to (C) with a on its boundary. Its centroid is farther away from a due to the removal of area on one side.

From now on we can restrict ourselves to symmetrical domains with diameter $\leqq \Delta$ having AB as part of their boundary. They must obviously lie entirely in the intersection N of the two semicircular regions mentioned in §4.

In such a region R we draw the parallel line p through G to AB. Let S and T be its intersections with the boundary of R.

(d) We intersect the lines AS and BT with the boundary of N in D and E and thus obtain in $ABECD$ a figure R' of class (Q) with greater y, since below p area is removed, and above p area is added.

(e) If finally the last region so obtained does not belong to (P) we can enlarge y by replacing the domain by a $P_{\max}$ according to the lemma in the preceding section.

7. **Inequalities.** Since both the endpoints of interval (8) are excluded the maximum value of (13) occurs at a zero of the derivative $\partial\eta/\partial\theta$. This leads to the following equation for $\theta(\alpha)$:

$$\begin{aligned}(23)\qquad &\theta\cos(\alpha+2\theta)+\frac{1}{4}\sin\alpha\cos^2\alpha\\ &\qquad=\sin\theta[\cos(\alpha+\theta)+\sin\alpha\sin(\alpha+\theta)\cos(\alpha+2\theta)].\end{aligned}$$

Writing $\tan\alpha=t$ and $\tan\theta=T$ it can be put into the form

$$(24)\qquad \operatorname{arc\,tg} T = R(t, T),$$

where $R(t, T)$ is the following rational function:

$$R(t, T)=\frac{4T(1+t^2)(1+T^2)(1-tT)+4tT(t+T)(1-T^2-2tT)-t(1+T^2)^2}{4(1+t^2)(1+T^2)(1-T^2-2tT)}.$$

Thus we proved the

THEOREM. *If the diameter D of a convex disc R satisfies $D\leqq\Delta$ then for the distance y of its centre of gravity from one of its chords of length $a\neq 0$ the sharp inequality*

$$(25)\qquad y\leqq\Delta\cdot\eta(\alpha)$$

holds, where $\sin\alpha=a/(2\Delta)$ *and* $\eta(\alpha)$ *is the result of eliminating* θ *from* (13) *and* (23). *The equality sign holds in* (25) *if and only if R is a pentagon of class* (P) *with an angle* $\theta(\alpha)$ *determined by* (23).

For $a=\alpha=0$ the same arguments used in the general case show that $y<2\Delta/3$, where the bound can be arbitrarily approximated by narrow triangular sectors. Through the base point A of a domain $P_{\max}$ for $\alpha\neq 0$ (see figure) there is a supporting line meeting the disc only in A and having a distance Y from the centroid greater than $\eta(\alpha)\Delta$. Applying the preceding remark to the chord AA of length 0 on this supporting line we find $\eta(\alpha)\Delta<Y<2\Delta/3$, hence for all α we have $\eta(\alpha)<2/3$. Thus we showed

COROLLARY I. *All chords and supporting lines of a convex disc of diameter Δ are closer to its centre of gravity than $2\Delta/3$, and this bound cannot be improved.*

In the other limit case $\alpha=\pi/6$ we can actually derive an equation for $\gamma=\eta(\pi/6)$. Making in (23) the substitution $\theta=\pi/6-\vartheta$ we arrive at the relation

$$(26)\quad 4\pi-3\cdot 3^{1/2}+12(\sin 2\vartheta-2\vartheta)=\frac{-9+48\cos(2\vartheta)-24\cos^2(2\vartheta)}{4\sin(2\vartheta)}.$$

Combining this with (13) we obtain

$$\gamma \equiv \eta\left(\frac{\pi}{6}\right) = \frac{2}{3}\sin(2\vartheta). \tag{27}$$

Solving (27) for ϑ ($0 \leqq \vartheta \leqq \pi/6$ and $0 \leqq \gamma \leqq 3^{1/2}/3$ because of (9)) and substituting into (26) leads to

$$\begin{aligned} 9\gamma^2 + (4\pi - 3\cdot 3^{1/2})\gamma + 11/2 - 12 \arcsin(3\gamma/2) \\ - 4\left|(4 - 9\gamma^2)^{1/2}\right| = 0. \end{aligned} \tag{28}$$

Since the left hand side of (28) is of different sign for $\gamma = 0$ and $\gamma = 2/3$ and its derivative always positive there is exactly one solution in this interval. By an iteration process it was found to be

$$\gamma \approx 0.3426 \cdots.$$

Summing up:

COROLLARY II. *Between the diameter Δ of a convex disc R and the distance y of its centre of gravity from any diameter the following sharp inequality holds*

$$y \leqq \gamma\Delta, \tag{29}$$

where $\gamma = 0.3426 \cdots$ is the solution of the transcendental equation (28) *in the range $0 < \gamma < 2/3$. The equality sign holds in* (29) *if and only if R is a pentagon of class (P) with base $\overline{AB} = \Delta$ and angle $ABE = \pi/2 - \vartheta = 74°32'$.*

It is worth remarking that if the condition of convexity is dropped the factor 0.3426 in (29) must be replaced by the much larger value $3^{1/2}/2 = 0.866$; for the centroid can come arbitrarily close to the highest point C in our figure.

8. **Numerical results.**[2] Using (24) the function $\eta(\alpha)$ occurring in (25) is tabulated below for a few values of $\lambda = a/\Delta$. Also given are the corresponding angles $\theta(\alpha)$ of the maximizing domains.

If $a_1 = a_2$, $\Delta_1 > \Delta_2$ or $\lambda_1 < \lambda_2$ it follows from our theorem that $\Delta_1\eta(\alpha_1) \geqq \Delta_2\eta(\alpha_2)$; it seems from the table, however, that even $\eta(\alpha_1) > \eta(\alpha_2)$ holds for any pair $\alpha_1 < \alpha_2$.

The table gives also the upper bounds $\theta_u(\alpha)$ from (8), corresponding to the domains of class (Q) with $L = DE = \Delta$, together with their values of η. It is curious to note that although $\theta(\alpha)$ and $\theta_u(\alpha)$ differ considerably the corresponding values of η are all close to each other.

[2] These computations were made possible by a grant from the University of Alberta.

Although the regions $P_{\max}$ must win according to our lemma (§5) the domains with $L=\Delta$ come very close to being the best ones.

TABLE

$\lambda = \frac{a}{\Delta} = 2 \sin \alpha$	$\theta(\alpha)$	$\eta(\alpha)$	$\theta_u(\alpha)$	$\eta[\alpha, \theta_u(\alpha)]$
0.0	0.00 000	0.66 67	0.26 18	0.63 66
0.2	0.22 151	0.57 07	0.27 09	0.56 81
0.4	0.24 969	0.50 96	0.28 70	0.50 82
0.6	0.25 904	0.45 35	0.31 13	0.45 13
0.8	0.25 938	0.39 84	0.35 41	0.39 40
1.0	0.25 368	0.34 26	0.52 36	0.33 92

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