MULTIVALENTLY CLOSE-TO-CONVEX FUNCTIONS

TOSHIO UMEZAWA

1. **Introduction.** The class of close-to-convex functions in $|z|<1$ introduced by Kaplan [1] is characterized by the property that there exists a function $\phi(z)$ analytic, schlicht and convex in $|z|<1$ for which $R[f'(z)/\phi'(z)]>0$, $|z|<1$. Such functions are necessarily schlicht.

Let

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots \tag{1.1}$$

be a function of this class. Then it has been pointed out by Reade [2] that the inequalities

$$|a_n| \leqq n, \qquad n = 2, 3, \cdots \tag{1.2}$$

hold. This result extends (1.2) to a wider class than was previously considered [3].

In this paper we extend these results to the case of functions p-valent in the unit circle.

2. **Preliminaries.** Hereafter we consider mainly functions of the form

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n, \qquad 1 \leqq q \leqq p. \tag{2.1}$$

Definition 1. The function $\phi(z)$ of the form (2.1) is said to be an element of the class $C(p)$, p a positive integer, if it is regular in $|z|<1$ and if there is a $\rho<1$ such that for all r in the interval $\rho<r<1$

$$G(r,\theta) = 1 + R\left(re^{i\theta}\frac{\phi''(re^{i\theta})}{\phi'(re^{i\theta})}\right) > 0, \qquad 0 \leqq \theta \leqq 2\pi, \tag{2.2}$$

and

$$\int_0^{2\pi} G(r,\theta)\,d\theta = 2\pi p. \tag{2.3}$$

This definition is due to Goodman [4]. It is shown in [4] that if $\phi(z)\in C(p)$, it maps $|z|\leqq r<1$ onto a locally convex region and that $\phi(z)$ is at most p-valent in $|z|<1$.

Received by the editors December 5, 1956 and, in revised form, January 31, 1957.

REMARK. Since

$$1 + R\frac{z\phi''(z)}{\phi'(z)} = \frac{d \arg d\phi(z)}{d \arg dz}$$

(2.2) is equivalent to

$$d \arg d\phi(z) > 0, \qquad |z| = r < 1 \tag{2.4}$$

and (2.3) means that $\phi(z)$ has $p-1$ critical points in $|z|<1$.

DEFINITION 2. The function $f(z)$ is said to be an element of the class $K(p)$, p a positive integer, if it is regular in $|z|<1$ and if there exists a function $\phi(z)$ of the class $C(p)$ for which $R[f'(z)/\phi'(z)]>0$, $|z|<1$ holds.

Such functions will be called close-to-convex p-valent functions. We shall prove that such functions are at most p-valent in $|z|<1$. For the purpose we need the following lemma due to the author [5].

LEMMA 1. *Let $f(z)$ be regular for a simply connected closed domain D whose boundary Γ consists of a regular curve and $f'(z)\neq 0$ on Γ. Suppose that*

$$\int_\Gamma d \arg df(z) = 2p\pi. \tag{2.5}$$

If we have for an arbitrary arc C on the boundary Γ

$$\int_C d \arg df(z) < (2p+1)\pi \tag{2.6}$$

or

$$\int_C d \arg df(z) > -\pi, \tag{2.7}$$

then $f(z)$ is at most p-valent in D.

To handle the coefficient problem and the deformation problem for the class $K(p)$ we need the following lemma due to [4] and [6].

LEMMA 2. *Let $\phi(z)\in C(p)$ of the form (2.1) have $p-q$ critical points $\beta_1, \beta_2, \cdots, \beta_{p-q}\neq 0$ in $|z|<1$. Then*

$$\phi'(z) \ll qF(z)/z \qquad \textit{for } |z| < 1 \tag{2.8}$$

and

$$|\phi'(re^{i\theta})| \leqq qF(r)/r \qquad \textit{for } r < 1 \tag{2.9}$$

where $F(z)$ is defined by

$$F(z) = \frac{z^q}{(1-z)^{2p}} \prod_{j=1}^{p-q} \left(1 + \frac{z}{|\beta_j|}\right)(1 + z|\beta_j|). \tag{2.10}$$

3. The main theorems.

THEOREM 1. *If $f(z) \in K(p)$, then $f(z)$ is at most p-valent in $|z| < 1$.*

PROOF. By Definition 2 there exists a function $\phi(z)$ of the class $C(p)$ such that, for $|z| < 1$,

$$R[f'(z)/\phi'(z)] = R[df(z)/d\phi(z)] > 0. \tag{3.1}$$

Since $\phi'(z)$ has exactly $p-1$ roots in $|z| < 1$ by Definition 1, $f'(z)$ must have the same roots as $\phi'(z)$ by (3.1). Hence we have

$$\int_{|z|=r} d \arg df(z) = 2p\pi \qquad \text{for } \rho < r < 1. \tag{3.2}$$

Since $df/d\phi \neq 0$ in $|z| < 1$ by (3.1), arg $(df/d\phi)$ is one-valued in $|z| < 1$. On the other hand we have

$$\arg df = \arg \frac{df}{d\phi} + \arg d\phi.$$

Hence arg df is also one-valued if we take a suitable branch of arg $d\phi(z)$.

By taking these branches and by the assumption (3.1), we have

$$-\pi/2 < \arg df(z_i) - \arg d\phi(z_i) < \pi/2$$

and

$$-\pi/2 < -\arg df(z_j) + \arg d(\phi z_j) < \pi/2$$

for arbitrary points z_i, z_j, $i > j$ on $|z| = r$, sufficiently near to 1. Hence we have

$$-\pi < \arg df(z_i) - \arg df(z_j) - (\arg d\phi(z_i) - \arg d\phi(z_j)) < \pi$$

where $2p\pi \geqq \arg d\phi(z_i) - \arg d\phi(z_j) > 0$ since $\phi(z) \in C(p)$. Thus we have

$$-\pi < \arg df(z_i) - \arg df(z_j) < (2p+1)\pi,$$

which is equivalent to

$$-\pi < \int_{z_j}^{z_i} d \arg df(z) < (2p+1)\pi. \tag{3.3}$$

(3.2) and (3.3) show that $f(z)$ is at most p-valent in $|z|<1$ by Lemma 1.

THEOREM 2. *Let*

$$f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \tag{3.4}$$

be a function of the class $K(p)$. Suppose that in addition to the $(q-1)$th order critical points at $z=0$ the function $f(z)$ has exactly $p-q$ critical points $\beta_1, \beta_2, \cdots, \beta_{p-q}$ such that $0<|\beta_j|<1$, $j=1, 2, \cdots, p-q$. Then

$$|a_n| \leqq C_n, \qquad n = q+1, q+2, \cdots, \tag{3.5}$$

$$|f(re^{i\theta})| \leqq F(r) \qquad \textit{for } r < 1 \tag{3.6}$$

and

$$|f'(re^{i\theta})| \leqq F'(r) \qquad \textit{for } r < 1 \tag{3.7}$$

where C_n and $F(r)$ are defined by

$$\begin{aligned} F(z) &= q\int_0^z \frac{t^{q-1}(1+t)}{(1-t)^{2p+1}} \sum_{j=1}^{p-1}\left(1+\frac{t}{|\beta_j|}\right)(1+t|\beta_j|)dt \\ &= z^q + \sum_{n=q+1}^{\infty} c_n z^n. \end{aligned} \tag{3.8}$$

PROOF. By our hypothesis, there exists a function of the class $C(p)$ which has the same form and the same critical points as $f(z)$ such that $R[f'(z)/\phi'(z)]>0$ for $|z|<1$. Hence by Carathéodory's theorem

$$\frac{f'(z)}{\phi'(z)} \ll \frac{1+z}{1-z}$$

and by Strohhäcker's theorem [7]

$$\frac{1-r}{1+r} \leqq \left|\frac{f'(re^{i\theta})}{\phi'(re^{i\theta})}\right| \leqq \frac{1+r}{1-r} \qquad \text{for } r < 1. \tag{3.9}$$

On the other hand we have (2.8) and (2.9) for $\phi'(z)$. Accordingly we have (3.7) and

$$f'(z) \ll \frac{z^{q-1}(1+z)}{(1-z)^{2p+1}} \prod_{j=1}^{p-q}\left(1+\frac{z}{|\beta_j|}\right)(1+z|\beta_j|).$$

Hence we have (3.5) and (3.6) with (3.8). q.e.d.

COROLLARY 1. *If* $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n \in K(p)$, *then* $f(z)$ *is* p-*valent for* $|z|<1$ *and the inequalities*

$$(3.10)\qquad |a_n| \leqq \binom{n+p-1}{2p-1}, \qquad n=p+1, p+2, \cdots,$$

$$(3.11)\qquad \frac{r^p}{(1+r)^{2p}} \leqq |f(re^{i\theta})| \leqq \frac{r^p}{(1-r)^{2p}} \qquad \text{for } r<1$$

and

$$(3.12)\qquad \frac{pr^{p-1}(1-r)}{(1+r)^{2p+1}} \leqq |f'(re^{i\theta})| \leqq \frac{pr^{p-1}(1+r)}{(1-r)^{2p+1}} \qquad \text{for } r<1$$

hold.

PROOF. (3.10) and the upper bounds in (3.11) and (3.12) are obvious. Let us prove the lower bounds in them.

In this special case, for the corresponding convex p-valent function $\phi(z)$, we have

$$\frac{pr^{p-1}}{(1+r)^{2p}} \leqq |\phi'(re^{i\theta})| \qquad r<1$$

since $z\phi'(z)$ is p-valently starlike in $|z|<1$, [4]. Hence we have the lower bound for $|f'(z)|$ in (3.12) from (3.9).

Let w be one of the nearest points from the origin to the image curve of $|z|=r<1$ under $f(z)$. Further let L be the curve on the z-plane corresponding to the line segment $0w$. Then we have

$$|f(z)| = \int_L |f'(z)|\,|dz| \geqq \int_0^{|z|} |f'(z)|\,d|z|$$

$$\geqq \int_0^{|z|} \frac{p|z|^{p-1}(1-|z|)}{(1+|z|)^{2p-1}}\,d|z| = \frac{|z|^p}{(1+|z|)^{2p}},$$

which gives the lower bound in (3.10).

But we remark that the inequalities (3.11) follow at once from the p-valency of $f(z)$ by the result due to W. K. Hayman [8].

COROLLARY 2. *Let* $f(z)=z^p+\sum_{n=p+1}^{\infty} a_n z^n$ *be regular for* $|z|<1$. *Suppose that* $f(z)$ *satisfies one of the following conditions*:

(1) $\qquad R[zf'(z)/f(z)]>0 \qquad$ *for* $|z|<1$.

(2) $\qquad R[f'(z)/z^{p-1}]>0 \qquad$ *for* $|z|<1$.

(3) $\qquad R[(1-z)^p f'(z)/z^{p-1}]>0 \qquad$ *for* $|z|<1$.

(4) $\qquad R[(1-z^p)f'(z)/z^{p-1}]>0 \qquad$ *for* $|z|<1$.

(5) $$R[(1-z)^{2p}f'(z)/z^{p-1}]>0 \qquad for\ |z|<1.$$

(6) $$R[(1-z^p)^2f'(z)/z^{p-1}]>0 \qquad for\ |z|<1.$$

(7) $$R[(1-z^2)^pf'(z)/z^{p-1}]>0 \qquad for\ |z|<1.$$

(8) $$R[(1-z^{2p})f'(z)/z^{p-1}]>0 \qquad for\ |z|<1.$$

Then $f(z)$ *is* p*-valent for* $|z|<1$, *and* (3.10), (3.11) *and* (3.12) *hold.*

Proof. Under the condition (1), $\phi(z)=p\int_0^z(f(z)/z)dz\in C(p)$ and $R[f'(z)/\phi'(z)]=R[zf'(z)/f(z)]/p>0$ for $|z|<1$. Hence $f(z)\in K(p)$.

In the other cases, $f(z)\in K(p)$, since the following functions

$$(2)\ z^p,\quad (3)\ \int_0^z\frac{z^{p-1}}{(1-z)^p}dz,\quad (4)\ \int_0^z\frac{z^{p-1}}{1-z^p}dz,\quad (5)\ \int_0^z\frac{z^{p-1}}{(1-z)^{2p}}dz,$$

$$(6)\ \int_0^z\frac{z^{p-1}}{(1-z^p)^2}dz,\quad (7)\ \int_0^z\frac{z^{p-1}}{(1-z^2)^p}dz,\quad (8)\ \int_0^z\frac{z^{p-1}}{1-z^{2p}}dz$$

are easily seen to be p-valently convex in $|z|<1$.

Hence our conclusion follows in every case by Corollary 1.

Here we note that in almost every case in Corollary 2 the inequalities can be sharpened considerably by direct use of Carathéodory's and Strohhäcker's Theorems.

References

1. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Mathematical Journal vol. 1 (1952) pp. 169–185.

2. M. Reade, *Sur une classe de fonctions univalentes*, C. R. Acad. Sci. Paris vol. 239 (1954) pp. 1758–1759.

3. M. S. Robertson, *Analytic functions starlike in one direction*, Amer. J. Math. vol. 58 (1936) pp. 465–472.

4. A. W. Goodman, *On the Schwarz-Christoffel transformation and p-valent functions*, Trans. Amer. Math. Soc. vol. 68 (1950) pp. 204–223.

5. T. Umezawa, *On the theory of univalent functions*, Tôhoku Math. J. vol. 7 (1955) pp. 212–223.

6. ———, *A class of multivalent functions with assigned zeros*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 813–820.

7. E. Strohhäcker, *Beiträge zur Theorie der schlichten Funktionen*, Math. Zeit. vol. 37 (1933) pp. 356–380.

8. W. K. Hayman, *Symmetrization in the theory of functions*, Technical Report No. 11, Contract N6-ORI-106, Task Order 5(NR-043-992), Office of Naval Research, Washington, D. C.

Gumma University, Maebashi, Japan