2. D. G. Northcott, *Ideal theory*, Cambridge Univ. Press, 1953.

3. D. G. Northcott and D. Rees, *A note on reductions of ideals with application to the generalized Hilbert function*, Proc. Cambridge Philos. Soc. vol. 50 (1954) pp. 353–359.

4. P. Samuel, *Algèbre locale*, Mémorial des Sciences Mathématiques no. 123, Paris, 1953.

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CONCERNING REAL VALUED MAPS OF THE n-SPHERE[1]

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A definition for the width of a closed curve, believed to be new, is used together with some standard homology theory, to prove the following theorem: *Let $f: S^n \to E^1$ be continuous, $0 \leqq d \leqq 2$, p the covering map $S^n \to P^n$. If $X_d = \{x \in S^n \mid$ there exists $y \in S^n$ with $\rho(x, y) = d$, $f(x) = f(y)\}$, then pX_d carries the nontrivial* mod 2 Čech $n-1$ *cycle of P^n.* This generalizes Theorem 2 of [4], already considerably extended in other directions by Bourgin [2] and Yang [5].

We will use the following notation: E^{n+1} = Euclidean $n+1$ space, ρ is the Euclidean metric, ω is the origin in E^{n+1}. $S^n = \{x \in E^{n+1} \mid \rho(x, \omega) = 1\}$. P^n is projective n space. $p_i: A_1 \times A_2 \to A_i$, $i = 1, 2$ will denote the projection. $\mathcal{H}_p(A) = p$ dimensional Čech homology group of A with coefficients the integers mod 2 (the only coefficients to be used here). $T^2 = S^1 \times S^1$, Δ = diagonal in T^2. In a space X, $\tilde{A} = X - A$, $\bar{A}$ = closure of A.

LEMMA 1. *Let $g: S^1 \to E^1$ be continuous, X a connected subset of T^2 such that either $p_1X = S^1$ or $p_2X = S^1$. Then there exists $(x, y) \in X$ such that $g(x) = g(y)$.*

PROOF. If $p_1X = S^1$, then there exists $(x_1, y_1) \in X$ such that $g(x_1) = \max_{x \in S^1} g(x)$, and $(x_2, y_2) \in X$ such that $g(x_2) = \min_{x \in S^1} g(x)$. Then if neither $(g(x_1), g(y_1))$ nor $(g(x_2), g(y_2))$ is on the diagonal in E^2, then they are on opposite sides. If $p_1X \neq S^1$, then we may make the above argument on the second coordinate.

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Let S be a metric space with metric ρ. Define, for a closed curve $\alpha: S^1 \to S$,

$$\rho_\alpha: T^2 \to E^1 \text{ by } \rho_\alpha(\theta, \theta') = \rho(\alpha(\theta), \alpha(\theta')).$$

Let $A_d = \{(\theta, \theta') \in T^2 | \rho_\alpha(\theta, \theta') \geqq d\}$. Then define the width of the curve α by $w(\alpha) = \text{l.u.b.} \{d | i_*: \mathcal{H}_1(A_d) \to \mathcal{H}_1(T^2) \text{ is not trivial}\}$ (i_* is induced by the inclusion $i: A_d \to T^2$). [If $\alpha: S^1 \to E^2$ is convex, then $\omega(\alpha)$ is the usual minimum distance between parallel support lines.]

LEMMA 2. *$B = \{d | i_*: \mathcal{H}_1(A_d) \to \mathcal{H}_1(T^2)$ is not trivial$\}$ is an interval in $[0, \infty)$ containing 0, or else B contains only 0.*

PROOF. If $0 \leqq d'' < d'$, then $A_{d''} \supset A_{d'}$. Hence if $d' \in B$, $\mathcal{H}_1(A_{d'}) \to \mathcal{H}_1(A_{d''}) \to \mathcal{H}_1(T^2)$ is nontrivial, so $d'' \in B$.

LEMMA 3. *If $0 < d \in B$, and $C_d = \{(\theta, \theta') \in T^2 | \rho_\alpha(\theta, \theta') = d\}$, then $\mathcal{H}_1(C_d) \to \mathcal{H}_1(T^2)$ has for image the diagonal element* $(1, 1)$.

PROOF. Since $d \in B$, $\mathcal{H}_1(A_d) \to \mathcal{H}_1(T^2)$ is not trivial. Since $d > 0$' $A_d \cap \Delta = \phi$. Since the nontrivial nondiagonal elements of $\mathcal{H}_1(T^2)$ have a nontrivial intersection with the diagonal element, the image of $\mathcal{H}_1(A_d)$ must be $(1, 1)$. Furthermore, since $\Delta \subset \text{Cl}(\tilde{A}_d)$, the image of $\mathcal{H}_1(\text{Cl}(\tilde{A}_d))$ in $\mathcal{H}_1(T^2)$ contains $(1, 1)$. Hence in the Mayer-Vietoris homology sequence

$$\cdots \to \mathcal{H}_1(BdA_d) \xrightarrow{i} \mathcal{H}_1(A_d) + \mathcal{H}_1(\text{Cl}(\tilde{A}_d)) \xrightarrow{j} \mathcal{H}_1(T^2) \to \cdots$$

if $\Gamma_1 \in \mathcal{H}_1(A_d)$ maps by inclusion into $(1, 1)$ and $\Delta_1 \in \mathcal{H}_1(A_d)$ maps by inclusion into $(1, 1)$, then $j(\Gamma_1 + \Delta_1) = 0$, so $\Gamma_1 + \Delta_1 = i\Gamma_1'$ for some $\Gamma_1' \in \mathcal{H}_1(BdA_d)$. Then the successive inclusions $\mathcal{H}_1(BdA_d) \to \mathcal{H}_1(C_d) \to \mathcal{H}_1(A_d) \to \mathcal{H}_1(T^2)$ take Γ_1' into $(1, 1)$.

Let $T: S^n \to S^n$ be the antipodal map, γ_1 a nonbounding T-invariant cycle (see [5] for definitions) of some T-invariant triangulation of S^n. Then it is easily seen that there is a $\gamma_1' T \sim \gamma_1$ with $|\gamma_1'|$ a simple closed curve, $|\gamma_1'| \subset |\gamma_1|$.

LEMMA 4. *There exists a map $\alpha: S^1 \to |\gamma_1'|$ such that $w(\alpha) = 2$.*

PROOF. Let x, $-x$ be antipodal points of $|\gamma_1'|$, β a 1:1 map of the arc $0 \leqq \theta \leqq \pi$ of S^1 onto one of the polygonal arcs of $|\gamma_1'|$ joining x to $-x$. Extend β to $\alpha: S^1 \to |\gamma_1'|$ by defining $\alpha(\theta) = \beta(\theta)$, $\alpha(\theta + \pi) = T \circ \beta(\theta)$ for $0 \leqq \theta \leqq \pi$. $D = \{(\theta, \theta + \pi) \in T^2 | \text{all } \theta \in S^1\}$ is a "straight line" in T^2 parallel to the diagonal, so $\mathcal{H}_1(D) \to \mathcal{H}_1(T^2)$ is not trivial. Furthermore, $\rho_\alpha(\theta, \theta + \pi) = 2$ for all θ, so $D = A_2$, $w(\alpha) = 2$.

LEMMA 5. *Let* $X_d = \{x \in S^n \mid$ *there exists* $y \in S^n$ *with* $\rho(x, y) = d$, $f(x) = f(y)\}$. *Then if* $0 \leqq d \leqq 2$, *and* $\gamma_1 T \nsim 0$, *then* $|\gamma_1| \cap X_d \neq \phi$.

PROOF. Let γ_1' be as in Lemma 4, $\alpha: S^1 \to |\gamma_1'|$ with $w(\alpha) = 2$. Then by Lemma 2 for $0 < d \leqq 2$, $d \in B$, hence by Lemma 3, the image of $\mathcal{H}_1(C_d)$ in $\mathcal{H}_1(T^2)$ is the diagonal element. C_d contains a continuum C_d' such that the image of $\mathcal{H}_1(C_d')$ in $\mathcal{H}_1(T^2)$ is the diagonal element, and thus $p_1 C_d' = S^1$. Therefore, by Lemma 1, there exists $(\theta, \theta') \in C_d$ such that $f \circ \alpha(\theta) = f \circ \alpha(\theta')$, using $g = f \circ \alpha$. Hence $\alpha(\theta)$ and $\alpha(\theta')$ are points of $|\gamma_1'| \cap X_d \subset |\gamma_1| \cap X_d$.

PROOF OF THEOREM. Let U be any open subcomplex of a triangulation K containing pX_d in P^n. By Lemma 5, no nonbounding 1-cycle of K is carried by $\bar{U}$. Hence, by the Pontrjagin Removing Theorem, [1] there exists a 1-cocycle γ^0, noncobounding on K and zero on $\bar{U}$. Hence, the dual $n-1$ cycle γ_{n-1} is nonbounding in K, and carried by the dual subdivision of U. By the continuity of the Čech theory [3], the theorem now follows.

REMARKS. One may obtain as a corollary the following: Let $f_1, \cdots, f_n$ be maps of $P^n \to E^1$, $d_1, \cdots, d_n$ numbers with $0 \leqq d_i \leqq 1$. Let P^n be metrized by $S(x, y) = \rho(p^{-1}x, p^{-1}y)$. Then there exist points $x_0, x_1, \cdots, x_n \in P^n$ such that $f_i(x_0) = f_i(x_i)$ and $S(x_0, x_i) = d_i$.

Notice that a considerable amount of information about X_d was not used in obtaining our theorem. While it was proved that every curve α of width $\geqq d$ must intersect X_d, we used only curves of width 2 for the theorem. To get stronger results, one may have to abandon homology methods.

BIBLIOGRAPHY

1. P. Alexandroff, *On homological situation properties of complexes and closed sets*, Trans. Amer. Math. Soc. vol. 54 (1943) p. 295.

2. D. G. Bourgin, *On some separation and mapping theorems*, Comment Math. Helv. vol. 29 (1955) pp. 199–214.

3. S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, 1952.

4. G. R. Livesay, *On a theorem of F. J. Dyson*, Ann. of Math. vol. 59 (1954) pp. 227–229.

5. C. T. Yang, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobo and Dyson*, I, Ann. of Math. vol. 60 (1954) pp. 262–282.

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