A NOTE ON REPRESENTATIONS OF INVERSE SEMIGROUPS

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It is known [1; 2] that every inverse semigroup $S$ has a faithful representation as a semigroup of $(1, 1)$-mappings of subsets of a set $A$ into $A$. The set $A$ may be taken as the set of elements of $S$ and the $(1, 1)$-mappings as mappings of principal left ideals of $S$ onto principal left ideals of $S$. If $E$ is the set of idempotents of $S$ then there is also a representation of $S$, not necessarily faithful, as a semigroup of $(1, 1)$-mappings of subsets of $E$ into $E$ [2]. If $e \in E$ denote by $S_e$ the subsemigroup $eSe$ of $S$. In this note we give a representation of any inverse semigroup $S$ as a semigroup of isomorphisms between the semigroups $S_e$. The representation is faithful if (a more general condition is given below) the center of each maximal subgroup of $S$ is trivial.

We recall that an inverse semigroup [3] is a semigroup $S$ in which for any $a \in S$ the equations $xax = x$ and $axa = a$ have a unique common solution $x \in S$ called the inverse of $a$ and denoted by $a^{-1}$ [5; 6]. This implies that the idempotents of $S$ commute and that to each $a \in S$ there corresponds a pair of idempotents $e, f$ such that $aa^{-1} = e$, $a^{-1}a = f$, $ea = a$, $af = a$. The idempotents $e, f$ are called respectively the left and right units of $a$. For any two elements $a, b \in S$, $(ab)^{-1} = b^{-1}a^{-1}$ (see [3]). Throughout what follows $S$ will denote an inverse semigroup and $E$ will denote its set of idempotents. If $e \in E$ then $S_e$ will denote the subsemigroup $eSe$ of $S$.

Lemma 1. If $e, f \in E$ then $S_e \cap S_f = S_{ef}$.

Proof. By Lemma 1 of [4] and its left-right dual $Se \cap Sf = Sef$ and $eS \cap fS = efS$. Hence since $S_e = eS \cap Se$ and $S_f = fS \cap Sf$, it follows that $S_e \cap S_f = efS \cap Sef = S_{ef}$.

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Lemma 2. If \( a \) is an element of \( S \) with left unit \( e \) and right unit \( f \) then \( S_e \) is isomorphic to \( S_f \).

Proof. We show that the mapping \( \alpha_a: s \rightarrow s\alpha_a = a^{-1}sa \), where \( s \in S_e \), is an isomorphism of \( S_e \) onto \( S_f \).

Let \( a^{-1}sa = t \); then since \( af = a \) and \( fa^{-1} = a^{-1} \), \( ft = t = tf \) and so \( t \in S_f \). Hence \( \alpha_a \) maps \( S_e \) into \( S_f \). Now let \( t \) be any element of \( S_f \). Then a repetition of the above argument with \( a^{-1} \) replacing \( a \) shows that \( ata^{-1} = s \in S_e \). Thus, since \( saa = a^{-1}sa = a^{-1}ata^{-1}a = ft = t \), the mapping \( \alpha_a \) is onto \( S_f \).

If \( a^{-1}s_1a = a^{-1}s_2a \) for \( s_1, s_2 \in S_e \) then \( aa^{-1}s_1aa^{-1} = aa^{-1}s_2aa^{-1} \), so that since \( aa^{-1} = e \) and \( es_i = s_e = s_i, \ s_1 = s_2 \). Hence \( \alpha_a \) is a \((1, 1)\)-mapping.

Finally that \( \alpha_a \) is a homomorphism follows because if \( s_1, s_2 \in S_e \), then \( s_1\alpha_a s_2\alpha_a = a^{-1}s_1aa^{-1}s_2a = a^{-1}s_1s_2a = (s_1s_2)\alpha_a \) because \( s_1 \) (or \( s_2 \)) \( \in S_e \) implies that \( s_1s_2 = s_1s_2 \).

The set of all elements of \( S \) with \( e \) as both left unit and right unit forms a group denoted by \( G_e \) [3]. The groups \( G_e \) are clearly the maximal subgroups of \( S \). We now have as a corollary to Lemma 2 the result

Corollary. If \( a \) is an element of \( S \) with left unit \( e \) and right unit \( f \) then \( G_e \) is isomorphic to \( G_f \).

Proof. It is easily seen that the restriction of \( \alpha_a \) to \( G_e \) maps \( G_e \) onto \( G_f \).

Denote by \( A(S) \) the set of isomorphisms \( \{ \alpha_a: a \in S \} \), defined in the proof of Lemma 2. Since an isomorphism is a \((1, 1)\)-mapping, the set \( A(S) \) generates a semigroup \( MA(S) \), say, of \((1, 1)\)-mappings formed by taking all finite products of the elements of \( A(S) \). If \( \alpha \) and \( \beta \) are \((1, 1)\)-mappings the product \( \alpha \beta \) is the mapping \( \alpha \) followed by the mapping \( \beta \) applied to those elements for which this sequence of mappings can be carried out [2]. If there are no such elements we write \( \alpha \beta = 0 \), and can regard 0 as the unique \((1, 1)\)-mapping of the elements of the empty set into the empty set. Then we have

Lemma 3. \( A(S) = MA(S) \).

Proof. It is sufficient to show that for any \( a, b \in S \), \( \alpha_a \alpha_b \in A(S) \).

Let \( aa^{-1} = e, a^{-1}a = f, bb^{-1} = g, b^{-1}b = h \), so that \( \alpha_a \) maps \( S_f \) onto \( S_f \) and \( \alpha_b \) maps \( S_g \) onto \( S_h \). Then, by Lemma 1, \( S_f \cap S_g = S_{fg} \) and so \( \alpha_a \alpha_b \) maps \( S_{fg} \alpha_a^{-1} \) onto \( S_{fg} \alpha_b \). We show that \( \alpha_a \alpha_b = \alpha_{ab} \).

Let \( (ab)(ab)^{-1} = k \) and \( (ab)^{-1}(ab) = l \). Let \( x \in S_{fg} \alpha_a^{-1} \), so that \( x \in S_{e} \) and hence \( aa^{-1}xaa^{-1} = x \), and also \( \alpha_a \in S_{fg} \) so that \( fga^{-1}xfg = a^{-1}xa \), from which it follows that \( a^{-1}xfga^{-1} = aa^{-1}xaa^{-1} = x \). But \( afga^{-1} = aa^{-1}abb^{-1}a^{-1} = ab(ab)^{-1} = k \), and so \( kxk = x \) and \( x \in S_k \). Conversely,
if \( x \in S_k \), then \( f(g(xa))f = fga^{-1}xa = fga^{-1}xg = a^{-1}(ab)(ab)^{-1}x(ab)(ab)^{-1}a = a^{-1}xkxa = a^{-1}xa = x\alpha_a \), and so \( x\alpha_a \in S_{fg} \). We similarly prove that \( S_{fg\alpha_b} = S_l \).

Thus \( \alpha_a\alpha_b \) maps \( S_k \) onto \( S_l \) and since \( x\alpha_a\alpha_b = (a^{-1}xa)\alpha_b = b^{-1}a^{-1}xab = (ab)^{-1}x(ab) = x\alpha_{ab} \) for \( x \in S_k \), \( \alpha_a\alpha_b = \alpha_{ab} \). This completes the proof of the lemma.

In [3] it was shown that the homomorphic image of an inverse semigroup is an inverse semigroup. If \( \mu: S \rightarrow T \) is a homomorphic mapping of \( S \) onto \( T \), then the kernel \( N \) of \( \mu \) is the inverse image under \( \mu \) of the set of idempotents of \( T \), and \( N \) is the union of its components, each component being the inverse image of a single idempotent of \( T \). It was shown in [3] that, given \( S \) and \( T \), the homomorphism \( \mu \) is determined by the components of the kernel of \( \mu \).

The center of a semigroup \( T \) is the set \( Z(T) = \{ z: z \in T, \forall t \in T, zt = tz \} \). \( Z(T) \) is clearly a subsemigroup of \( T \). Denote by \( Z_e \) the center of the maximal subgroup \( G_e \) of \( S \). Then we have the

**Theorem.** The mapping \( \mu: S \rightarrow A(S) \) of \( S \) onto \( A(S) \) defined by \( a\mu = \alpha_a \) for \( a \in S \) is a homomorphism. The kernel of \( \mu \) is \( N = \bigcup \{ N_e \} \), where \( N_e \) is the normal subgroup \( Z(S_e) \bigcap Z_e \) of \( G_e \) and the union is taken over all \( e \in E \).

**Proof.** We have already seen in the course of the proof of Lemma 3 that \( \mu \) is a homomorphism of \( S \) onto \( A(S) \).

It remains to determine the kernel of \( \mu \). Let \( \alpha_a \) be an idempotent of \( A(S) \). Then \( \alpha_a \) must be the identical mapping of some set \( S_e \), and so for \( s \in S_e \), \( s\alpha_a = a^{-1}sa = s \). Hence \( aa^{-1}sa = as \), and since \( aa^{-1} = e \), \( aa^{-1}s = s \), therefore \( sa = as \), and so \( a \in Z(S_e) \). Since also \( a \in G_e \), and \( G_e \subseteq S_e \), therefore \( a \in Z_e \). Hence \( a \in Z(S_e) \bigcap Z_e \).

Conversely, let \( b \in Z(S_e) \bigcap Z_e \). Then \( b \in G_e \) so that \( bb^{-1} = e = b^{-1}b \), and \( bs = sb \) for all \( s \in S_e \). Hence \( s\alpha_b = b^{-1}sb = b^{-1}bs = es = s = s\alpha_a \), so that \( \alpha_a = \alpha_b \).

Thus \( N_e = \mu^{-1}(\alpha_a) = Z(S_e) \bigcap Z_e \); which completes the proof of the theorem.

**Corollary.** If for each \( e \in E \), \( Z(S_e) \bigcap Z_e = e \), then the mapping \( \mu \) is an isomorphism.

It follows, as we remarked earlier, that if each \( Z_e = e \), that is if the center of each maximal subgroup of \( S \) is trivial, then \( S \) has a faithful representation as a semigroup of isomorphisms between the semigroups \( S_e \).

Finally we remark that in this latter case when each \( Z_e = e \), the isomorphisms \( \alpha_a \) are determined by their restrictions to the groups \( G_e \).
Thus we can regard $A(S)$ as a semigroup of isomorphisms between the groups $G_e$. When the elements of $A(S)$ are so regarded the product of two elements of $A(S)$ cannot be defined in a natural way independently of $S$.

REFERENCES


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