THE DEFINITION OF UNIVERSAL TURING MACHINE

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In [1], the author has proposed a definition of the notion, "universal Turing machine." The definition in [1] is open to the objection that a Turing machine may qualify as universal, although many computations (runs) are required to produce a single answer. In the present note, we propose an alternative definition which is free of this objection. However, it then turns out that whereas a Turing machine which is universal in the new sense also is universal in the sense of [1], it is easy to construct a Turing machine which is universal in the sense of [1], but not in the new sense.

We shall say that a Turing machine Z is *universal* (I), if it is universal in the sense of Definition II.6 of [1].

DEFINITION 1. For each (simple[1]) Turing machine Z, we write $\Phi_Z(\alpha)=\beta$ if there exists a sequence $\alpha=\alpha_1, \alpha_2, \cdots, \alpha_n=\beta$ of instantaneous descriptions[1] such that $\alpha_i \rightarrow \alpha_{i+1}(Z)$, $i=1, 2, \cdots, n-1$, where β is terminal.[1] If no such sequence exists, for a particular instantaneous description α, then $\Phi_Z(\alpha)$ remains undefined.

DEFINITION 2. A Turing machine M is *universal* (II) if there exist recursive[2] functions $g(\alpha)$ (defined on the set of instantaneous descriptions, but with numerical values) and $\rho(z, x)$ (defined on the set of pairs of non-negative integers, but with instantaneous descriptions as values) such that[3]

$$U\left(\min_y T(z, x, y)\right) = g(\Phi_M(\rho(z, x))).$$

To encode the computation of an arbitrary partially computable function $\psi_Z(x)$ via a Turing machine M which is universal (II), we let z_0 be a Gödel number of Z and note that (by the normal form theorem):

$$\begin{aligned}\psi_Z(x) &= U\left(\min_y T(z_0, x, y)\right)\\ &= g(\Phi_M(\rho(z_0, x))).\end{aligned}$$

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[1] For terminology, cf. [2].

[2] Here, in order to make our references to recursive functions refer to numerical functions it would be necessary to make use of Gödel numbers of instantaneous descriptions in the familiar manner.

[3] Equality of partial functions is to be taken as including the equality of their domains of definition.

THEOREM 1. *If M is universal* (II), *then M is universal* (I).

PROOF. Let M be universal (II). Using the notation of [1], and writing "gn" for "Gödel number of," we have:

$$\begin{aligned} U_n &= \left\{x \,\middle|\, \bigvee_y T(n, x, y)\right\} \\ &= \left\{x \,\middle|\, U\left(\min_y T(n, x, y)\right) \text{ is defined}\right\} \\ &= \{x \mid g(\Phi_M(\rho(n, x))) \text{ is defined}\} \\ &= \{x \mid \Phi_M(\rho(n, x)) \text{ is defined}\} \\ &= \{x \mid \rho(n, x) \in D_M\} \\ &= \{x \mid \text{gn } \rho(n, x) \in \delta_M\}. \end{aligned}$$

Hence, M is universal (I).

That the converse of Theorem 1 is false follows at once from the fact that a Turing machine M may be universal (I), although $\Phi_M(\alpha)$ is a constant on D_M, its domain of definition.

The present encoding has the three properties listed below Definition II.6 of [1]. This is easily seen using Theorem V.3 of [1].

REFERENCES

1. M. D. Davis, *A note on universal Turing machines.* Automata Studies, Annals of Mathematics Studies, Princeton University Press, 1956.

2. Martin Davis, *Computability and unsolvability,* McGraw-Hill and Company, to appear 1958.

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