HALF-HOMOMORPHISMS OF GROUPS

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Let G and G' be multiplicative systems. A *half-homomorphism* of G into G' will mean a mapping $a \rightarrow a'$ of G into G' such that for all $a, b \in G$, $(ab)' = a'b'$ or $b'a'$. An *anti-homomorphism* is a mapping such that always $(ab)' = b'a'$. The terms *half-isomorphism*, etc., are defined similarly.

It will be shown that any half-homomorphism of a group G into a group G' is either a homomorphism or an anti-homomorphism (Theorem 2). The corresponding theorem for nonassociative rings (with the added requirement that $(a+b)' = a' + b'$) was proved by Hua [**1**] (see also Jacobson and Rickart [**2**, Lemma 1]). For half-isomorphisms, it is sufficient to assume that G and G' are cancellation semigroups in order to obtain the analogous result (Theorem 1). Examples are given to show that Theorem 1 is false for semi-groups and for loops.

THEOREM 1. *Every half-isomorphism of a cancellation semi-group G into a cancellation semi-group G' is either an isomorphism or an anti-isomorphism.*

PROOF. The proof will consist of six steps.[2]

(1) If $xy = yx$, then $x'y' = y'x'$.

PROOF. By proper choice of notation, $(xy)' = x'y'$. Hence $(x(xy))' = x'^2y'$ or $x'y'x'$, while $(x^2y)' = x'^2y'$ or $y'x'^2$. If $(x^2y)' \neq x'^2y'$, then $x'y'x' = y'x'^2$ and $x'y' = y'x'$. If, on the other hand, $(x^2y)' = x'^2y'$, then $(x^2y \cdot y)' = x'^2y'^2$ or $y'x'^2y'$, but $(x^2y^2)' = ((xy)^2)' = x'y'x'y'$, and so in either case, $x'y' = y'x'$.

(2) If $(xy)' = x'y'$, then $(yx)' = y'x'$. If $(xy)' = y'x'$, then $(yx)' = x'y'$.

PROOF. Let $(xy)' = x'y'$. If $yx = xy$, then $(yx)' = (xy)' = x'y' = y'x'$ by (1). If $yx \neq xy$, then the assertion follows since the mapping is 1-1. The case where $(xy)' = y'x'$ is similar.

(3) $(xyx)' = x'y'x'$.[3]

PROOF. If $xy = yx$, then (3) follows from (1). If $xy \neq yx$, then by (2), either (i) $(x^2y)' = x'^2y'$ and $(yx^2)' = y'x'^2$, or (ii) $(x^2y)' = y'x'^2$ and

Presented to the Society, November 23, 1956; received by the editors July 13' 1956 and, in revised form, March 28, 1957.

[1] National Science Foundation Fellow.

[2] The author wishes to thank I. N. Herstein for suggesting step (3) in this proof.

[3] That is, a half-isomorphism is, in this case, a semi-isomorphism (see Dinkines [**3**]). The converse is not true even for groups since there exist semi-automorphisms which are neither automorphisms nor anti-automorphisms.

$(yx^2)'=x'^2y'$. In either case, since $x^2y\neq xyx\neq yx^2$, we must have $(xyx)'=x'y'x'$.

(4) If the theorem is false, then there exist x, y, $z\in G$ such that $(xy)'=x'y'\neq y'x'$ and $(xz)'=z'x'\neq x'z'$.

PROOF. Let A be the set of $x\in G$ such that $(xy)'=x'y'$ for all $y\in G$, and B the set of $x\in G$ such that $(xy)'=y'x'$ for all $y\in G$. If $A=G$, then the mapping is an isomorphism; if $B=G$, it is an anti-isomorphism. If there exists an $x\in G$ such that $x\notin A$ and $x\notin B$, then (4) is evident. Therefore, it may be supposed that $A<G,B<G$, $A\cup B=G$. Then there exist x, y, u, $v\in G$ such that $(xy)'=x'y'\neq y'x'$ and $(uv)'=v'u'\neq u'v'$. Thus by (2), $x\in A$, $y\in A$, $u\in B$, $v\in B$. If $xu\in A$, then

$$x'v'u' = x'(uv)' = (x(uv))' = ((xu)v)' = (xu)'v' = x'u'v',$$

so that $v'u'=u'v'$, a contradiction. Next note that $u'y'=(yu)'=y'u'$. If $xu\in B$, then

$$y'x'u' = y'(xu)' = (y(xu))' = (xu)'y' = x'u'y' = x'y'u'.$$

Hence $y'x'=x'y'$, again a contradiction. This proves (4).

Now deny the theorem and let x, y, z be as in (4).

(5) $z'x'y'=y'x'z'$.

PROOF. CASE 1. $yz\neq zy$. Then $(y(xz))'=y'z'x'$ or $z'x'y'$, while $((yx)z)'=y'x'z'$ or $z'y'x'$. But the cancellation law shows that $(yxz)'=y'z'x'$ is impossible, as is $(yxz)'=z'y'x'$. Hence (5) follows.

CASE 2. $yz=zy$. Then by (1), $(yz)'=y'z'=z'y'$, and $(x(yz))'=x'y'z'$ or $y'z'x'$, $((xy)z)'=x'y'z'$ or $z'x'y'$. Hence $(xyz)'=z'x'y'$ is impossible, so that $(xyz)'=x'y'z'$. Again $(y(zx))'=y'x'z'$ or $x'z'y'$, $((yz)x)'=y'z'x'$ or $x'y'z'$. Hence $(yzx)'=y'x'z'$ is impossible, and $(yzx)'=x'z'y'=x'y'z'=(xyz)'$. By (2), since $(x(yz))'=x'(yz)'$, also $(yzx)'=(yz)'x'=y'z'x'$. Hence $x'y'z'=y'z'x'$. Now $(y(xz))'=y'z'x'$ or $z'x'y'$. Since $yxz\neq yzx$ and $(yzx)'=y'z'x'$,

$$(yxz)' = z'x'y',\ ((yx)z)' = y'x'z' \text{ or } z'y'x'.$$

But $z'y'x'\neq z'x'y'$, hence $z'x'y'=y'x'z'$, and (5) holds.

(6) The theorem is true.

PROOF. Deny. Then by (3)

$$((xyx)z)' = x'y'x'z' \text{ or } z'x'y'x',$$

$$((xy)(xz))' = x'y'z'x' \text{ or } z'x'x'y'.$$

But $x'y'x'z'\neq x'y'z'x'$, and $z'x'y'x'\neq z'x'x'y'$. Also by (5), $x'y'x'z'=x'z'x'y'\neq z'x'x'y'$, $z'x'y'x'=y'x'z'x'\neq x'y'z'x'$. This is a contradiction, and therefore the theorem is true.

THEOREM 2. *Every half-homomorphism of a group G into a group G' is either a homomorphism or an anti-homomorphism.*

PROOF. Let N be the kernel of the given half-homomorphism. It follows easily that N is a normal subgroup of G. Let σ be the induced mapping from G/N into G': $(xN)\sigma = x'$. Then σ is a half-isomorphism of G/N into G'. By Theorem 1, σ is either an isomorphism or an anti-isomorphism. Hence the given half-homomorphism is either a homomorphism or an anti-homomorphism.

EXAMPLE 1. An example will be given of a semi-group S and a half-automorphism of S which is neither an automorphism nor an anti-automorphism. Let the elements of S be 1, 2, 3, 4, 5, 6, 7, 8, let $21 = 3$, $45 = 6$, $54 = 7$, and let all other products be 8. Then S is associative since any product of three elements is 8. Let $6' = 7$, $7' = 6$, and $i' = i$ otherwise. It is easily verified that this mapping is a half-automorphism, and that $(21)' = 2'1' \neq 1'2'$, $(45)' = 5'4' \neq 4'5'$. Thus the mapping is neither an automorphism nor an anti-automorphism of S.

It is perhaps worth noting that a unit element may be adjoined to S, and the half-automorphism extended to the enlarged semi-group.

EXAMPLE 2. An example will be given of a loop L and a half-automorphism of L which is neither an automorphism nor an anti-automorphism. Let L be the loop with elements $1, 2, \cdots, 8$, and multiplication table:

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	3	8	1	6	5	4	7
3	3	7	1	8	2	4	5	6
4	4	6	7	2	1	8	3	5
5	5	1	2	6	3	7	8	4
6	6	5	4	7	8	1	2	3
7	7	8	6	5	4	3	1	2
8	8	4	5	3	7	2	6	1

Let $7' = 8$, $8' = 7$, and $i' = i$ otherwise. It may then be verified that the mapping is a half-automorphism of L. However, $(24)' = 2'4' \neq 4'2'$, $(56)' = 6'5' \neq 5'6'$. Hence the mapping has the desired properties.

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A NOTE ON REPRESENTATIONS OF INVERSE SEMIGROUPS

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It is known [1; 2] that every inverse semigroup S has a faithful representation as a semigroup of (1, 1)-mappings of subsets of a set A into A. The set A may be taken as the set of elements of S and the (1, 1)-mappings as mappings of principal left ideals of S onto principal left ideals of S. If E is the set of idempotents of S then there is also a representation of S, not necessarily faithful, as a semigroup of (1, 1)-mappings of subsets of E into E [2]. If $e \in E$ denote by S_e the subsemigroup eSe of S. In this note we give a representation of any inverse semigroup S as a semigroup of isomorphisms between the semigroups S_e. The representation is faithful if (a more general condition is given below) the center of each maximal subgroup of S is trivial.

We recall that an inverse semigroup [3] is a semigroup S in which for any $a \in S$ the equations $xax = x$ and $axa = a$ have a unique common solution $x \in S$ called the inverse of a and denoted by a^{-1} [5; 6]. This implies that the idempotents of S commute and that to each $a \in S$ there corresponds a pair of idempotents e, f such that $aa^{-1} = e$, $a^{-1}a = f$, $ea = a$, $af = a$. The idempotents e, f are called respectively the left and right units of a. For any two elements $a, b \in S$, $(ab)^{-1} = b^{-1}a^{-1}$ (see [3]). Throughout what follows S will denote an inverse semigroup and E will denote its set of idempotents. If $e \in E$ then S_e will denote the subsemigroup eSe of S.

LEMMA 1. *If $e, f \in E$ then $S_e \cap S_f = S_{ef}$.*

PROOF. By Lemma 1 of [4] and its left-right dual $Se \cap Sf = Sef$ and $eS \cap fS = efS$. Hence since $S_e = eS \cap Se$ and $S_f = fS \cap Sf$, it follows that $S_e \cap S_f = efS \cap Sef = S_{ef}$.

Received by the editors March 9, 1957.