ON MONOTONE CONVERGENCE TO SOLUTIONS OF
\[ u' = g(u, t) \]

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1. Introduction. The purpose of this brief note is to show that Newton's method of successive approximation to the solution of a functional equation yields a monotone increasing sequence of approximations to the solution of the first order differential equation,

\[ u' = g(u, t), \quad u(0) = c, \]

provided that \( g(u, t) \) is uniformly convex in \( u \) or \( t \) in some fixed interval \([0, t_0]\).

The connection between the methods and results presented here and the theory of dynamic programming is treated in [1].

2. Newton's method of approximation. Write

\[ h(u, t) = \frac{\partial g}{\partial u}(u, t), \]

and consider the system of equations

\[ \begin{align*}
    \frac{du_0}{dt} &= g(v_0, t) + (u_0 - v_0)h(v_0, t), \quad u_0(0) = c, \\
    \frac{du_{n+1}}{dt} &= g(u_n, t) + (u_{n+1} - u_n)h(u_n, t), \quad u_{n+1}(0) = c,
\end{align*} \]

\[ n = 0, 1, 2, \ldots, \]

where \( v_0(t) \) is a known function of \( t \) which is taken to be continuous over \([0, t_0]\).

This is Newton's method of approximation applied to the differential equation of (1.1).

3. Monotonicity of convergence. Let us now demonstrate the following result

**Theorem.** Let \( h(u, t) \) exist, be continuous, and monotone increasing in \( u \) for \( 0 \leq t \leq t_0 \). Then

\[ u_0(t) \leq u_1(t) \leq \cdots \leq u_n(t) \leq \cdots, \quad 0 \leq t \leq t_1, \]

where \( t_1 > 0 \). The limit \( u(t) = \lim_{n \to \infty} u_n(t) \) is the solution of (1.1).

**Proof.** We begin with the observation that the uniform convexity

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of \( g(u, t) \) yields the result that

\[
(2) \quad g(u, t) = \max_v \left[ g(v, t) + (u - v)h(v, t) \right]
\]

for \( 0 \leq t \leq t_0 \).

Hence

\[
\frac{du_{n+1}}{dt} = g(u_n, t) + (u_{n+1} - u_n)h(u_n, t)
\]

\[
\leq g(v, t) + (u_{n+1} - v)h(v, t),
\]

where \( v = v(t) \) is the function which maximizes the function \( g(u, t) + (u_{n+1} - u)h(u, t) \). It follows that the solution of

\[
(4) \quad \frac{dw}{dt} = g(v, t) + (w - v)h(v, t), \quad w(0) = c,
\]

majorizes the solution of

\[
(5) \quad \frac{du_{n+1}}{dt} = g(u_n, t) + (u_{n+1} - u_n)h(u_n, t), \quad u_{n+1}(0) = c,
\]

within a common interval of existence, i.e. \( w(t) \geq u_{n+1}(t) \).

Since the function \( v(t) \) which maximizes is \( u_{n+1}(t) \), we see that the solution of (4) is precisely the function \( u_{n+2}(t) \).

This argument showed that \( U_M(t) \geq u_{0}(t) \) and, inductively, that (3.1) holds within a common interval of existence. That such an interval exists follows the usual lines, and similarly for the proof of convergence which is now equivalent to uniform boundedness of the sequence \( \{u_n(t)\} \).

4. Multi-dimensional case. The proof presented above hinged on the fact that any function \( u(t) \) satisfying the inequality

\[
(1) \quad \frac{du}{dt} \leq a(t)u + b(t), \quad u(0) = c,
\]

is majorized by the solution of

\[
(2) \quad \frac{dv}{dt} = a(t)v + b(t), \quad v(0) = c.
\]

The corresponding result for systems is not unreservedly true. If \( \{x_i(t)\} \) is a set of functions satisfying

\[
(3) \quad \frac{dx_i}{dt} \leq \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t), \quad x_i(0) = c_i, \quad i = 1, 2, \ldots, n,
\]
it is not necessarily true that \( x_i(t) \leq y_i(t) \), \( i = 1, 2, \ldots, n \), \( t \geq 0 \), where the \( y_i(t) \) satisfy the equations

\[
\frac{dy_i}{dt} = \sum_{j=1}^{n} a_{ij}(t)y_j + b_i(t), \quad y_i(0) = c_i, \quad i = 1, 2, \ldots, n.
\]

If, however, we have

\[
a_{ij}(t) \geq 0, \quad t \geq 0, \quad i \neq j,
\]

then the result does hold, cf. [2].

Consequently, if we take the Newton approximations

\[
\begin{align*}
\frac{du_{n+1}}{dt} &= f(u_n, v_n) + (u_{n+1} - u_n) \frac{\partial f}{\partial u_n} + (v_{n+1} - v_n) \frac{\partial f}{\partial v_n}, \quad u_{n+1}(0) = c_1, \\
\frac{dv_{n+1}}{dt} &= g(u_n, v_n) + (u_{n+1} - u_n) \frac{\partial g}{\partial u_n} + (v_{n+1} - v_n) \frac{\partial g}{\partial v_n}, \quad v_{n+1}(0) = c_2,
\end{align*}
\]

\( n = 0, 1, 2, \ldots \), with \( u_0, v_0 \) prescribed continuous functions, to the system

\[
\begin{align*}
\frac{du}{dt} &= f(u, v), \quad u(0) = c_1, \\
\frac{dv}{dt} &= g(u, v), \quad v(0) = c_2,
\end{align*}
\]

we can assert that

\[
u_n \leq u_{n+1}, \quad v_n \leq v_{n+1}, \quad t \geq 0, \quad n = 0, 1, 2, \ldots,
\]

provided that

\[
\frac{\partial f}{\partial v} \geq 0, \quad \frac{\partial g}{\partial u} \geq 0,
\]

for all \( u \) and \( v \), and provided that \( f(u, v) \) and \( g(u, v) \) are strictly convex functions of \( u \) and \( v \).

**Bibliography**


**Rand Corporation**