ON COHOMOLOGY OF LIE ALGEBRAS

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Let $G$ be a Lie algebra, $L$ an ideal of $G$ and $M$ a $G$-module. The $n$-dimensional cochains for $L$ in $M$ are $n$-linear alternating functions on $L^n$ with values in $M$. These cochains form a vector space $C^n(L, M)$ over the ground field of $L$; we identify $C^0(L, M)$ with $M$. $C^0(L, M)$ is already a $G$-module and for $n>0$, $f$ in $C^n(L, M)$ and $\gamma$ in $G$ put

$$(\gamma \cdot f)(\sigma_1, \ldots, \sigma_n) = \gamma \cdot f(\sigma_1, \ldots, \sigma_n) = \sum_{i=1}^{n} f(\sigma_1, \ldots, \sigma_{i-1}, [\gamma, \sigma_i], \sigma_{i+1}, \ldots, \sigma_n)$$

for all $\sigma_1, \ldots, \sigma_n$ in $L$.

This definition makes $C^n(L, M)$ into a $G$-module and further $\gamma \cdot (\delta f) = \delta(\gamma \cdot f)$ where $\delta$ is the coboundary operator: $C^n(L, M) \rightarrow C^{n+1}(L, M)$ so that the cohomology groups $H^n(L, M)$ take on the structure of a $G$-module. Professor Hochschild suggested to the author that he define an operation of $G$ on the standard interpretations of the low dimensional cohomology groups which would agree with the operation of $G$ on $H^n(L, M)$ as given above.

This problem has an analogue in the cohomology of groups. There, if $L$ is a normal subgroup of a group $G$ and $f$ is any function of $L^{n+1}$ to $M$, the operation

$$(\gamma \cdot f)(\sigma_0, \ldots, \sigma_n) = \gamma \cdot f(\gamma^{-1}\sigma_0\gamma, \ldots, \gamma^{-1}\sigma_n\gamma)$$

for $\gamma$ in $G$ and $\sigma_0, \ldots, \sigma_n$ in $L$, again induces the structure of a $G$-module on the cohomology groups $H^n(L, M)$. In an appendix we outline briefly a way of making the standard interpretations of the low dimensional cohomology groups into $G$-modules in a way consistent with the operation of $G$ on the cohomology groups.

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1. Let $G$ be a Lie algebra over a ground field of characteristic 0, $L$ an ideal of $G$ such that $G/L$ is semi-simple and let $M$ be a $G$-module. We denote by $\rho_n$ the restriction homomorphism of $H^n(G, M)$ into $H^n(L, M)$. For any $G$-module $U$, let $U^G = \{ u | \gamma \cdot u = 0 \text{ for all } \gamma \text{ in } G \}$. A special case of a theorem of Hochschild-Serre [1, p. 603, Theorem 13] is

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Numbers in square brackets refer to the bibliography at the end of the paper.
1.1. Theorem. If $G/L$ is semi-simple then $\rho_n$ is an isomorphism of $H^n(G, M)$ onto $H^n(L, M)^G$ for $n = 0, 1, 2$.

Hochschild-Serre's proof of their Theorem 13 depends on the theory of spectral sequences, and it seems desirable to sketch a direct proof of 1.1 here. Accordingly let $G = K + L$ be a Levi decomposition of $G$.

(1.2) $H^n(L, M)^G = H^n(L, M)^K$ for all $n \geq 0$.

(1.3) Every element of $H^n(L, M)^K$ has a representative cocycle in $C^n(L, M)^K$.

(1.4) If $df \in C^n(L, M)^K$ for $f$ in $C^{n-1}(L, M)$ then $\exists g$ in $C^{n-1}(L, M)^K$ s.t. $df = dg$.

(1.5) Let $f$ be in $Z^n(G, M)$ where $n = 1$ or $2$ then there exists $f^*$ in $Z^n(G, M) \ni f^*$ is cohomologous to $f$ and

(a) $f^*(k) = 0$ for all $k$ in $K$ (if $n = 1$),

(b) $f^*(\gamma_1, \gamma_2) = 0$ if $\gamma_1$ or $\gamma_2$ is in $K$ (if $n = 2$).

Proof. (a) If $f \in Z^1(G, M)$ then $f|_K \in Z^1(K, M)$ whence $\exists m$ in $C^0(K, M) = M$ s.t. $f(k) = k \cdot m$ for all $k$ in $K$. Put $f^*(k) = (f - dm)(k)$ for all $k$ in $G$.

(b) If $f \in Z^2(G, M)$, define a Lie algebra extension $(E, \pi)$ of $M$ by $G$ as follows: $E = (G, M)$ (direct product) as a vector space with $[(y_1, m_1), (y_2, m_2)] = ([\gamma_1, \gamma_2], \gamma_1 \cdot m_2 - \gamma_2 \cdot m_1 + f(\gamma_1, \gamma_2))$ for all $\gamma_1, \gamma_2$ in $G$, $m_1, m_2$ in $M$; $\pi$ the projection of $E$ onto $G$.

Now $G = K + L$ with $K$ semi-simple. $\pi^{-1}(L)$ is an ideal of $E$ containing $M$ and $E/\pi^{-1}(L) \cong K$. Thus we can find a linear map $\sigma: G \to E \ni \pi \sigma(\gamma) = \gamma$ for all $\gamma$ in $G$ and $\sigma|_K$ is a Lie algebra isomorphism of $K$ into $E$. Consider $\pi^{-1}(L)$ and $M$ as $\sigma(K)$-modules in the regular representation. Then since $\sigma(K)$ is semi-simple, there exists a submodule $H$ of $\pi^{-1}(L)$ s.t. $\pi^{-1}(L) = H + M$ and $H \cap M = (0)$. $\pi|_H$ is a vector space isomorphism of $H$ onto $L$ so we see that we can choose the linear map $\sigma$ so that

(1) $\sigma(k_1, k_2) = [\sigma(k_1), \sigma(k_2)]$ for all $k_1, k_2$ in $K$.

(2) $E = \sigma(K) + \sigma(L) + M$ direct sum as $\sigma(K)$-modules. Now note that $[\sigma(k), \sigma(x)] - \sigma[k, x] \subseteq \sigma(L) \to M = (0)$ for all $k$ in $K$, $x$ in $L$.

The cocycle $f^* \ni f^*(\gamma_1, \gamma_2) = [\sigma(\gamma_1), \sigma(\gamma_2)] - \sigma(\gamma_1, \gamma_2)$ is cohomologous to $f$ and has the desired properties.

Proof of 1.1. If $n = 0$ this is trivial. If $n = 1$ or $2$ and $h \in H^n(G, M)$, choose a representative cocycle $f$ of $h$ as in (1.5). It is easy to see that $k \cdot f|_L = 0$ for all $k$ in $K$ where $f|_L$ is the restriction of $f$ to $L$ so that

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2 (1.2), (1.3) and (1.4) are proved directly in [1, p. 603].
\[ \rho_n(H^n(G, M)) \subseteq H^n(L, M)^G. \] Using (1.4) one sees quickly that \( \rho_n \) is an isomorphism.

To see that \( \rho_n \) is onto, take \( \tilde{h} \) in \( H^n(L, M)^G \) \((n = 1, 2)\) and choose a representative \( f \) of \( \tilde{h} \) in \( Z^n(L, M)^K \) by (1.2) and (1.3). Extend \( f \) to \( \tilde{f} \) in \( Z^n(G, M) \) by

\[
\begin{align*}
n = 1 & \quad \tilde{f}(k + x) = f(x) \quad \text{for all } x \text{ in } L, k \text{ in } K; \\
n = 2 & \quad \begin{cases} 
\tilde{f}(k, \gamma) = 0 & \text{for all } \gamma \text{ in } G, k \text{ in } K, \\
\tilde{f}(x_1, x_2) = f(x_1, x_2) & \text{for all } x_1, x_2 \text{ in } L
\end{cases}
\end{align*}
\]

then \( \rho_n(\tilde{f} + dC^{-1}(G, M)) = \tilde{h} \).

1.6. Definition. Let \( G \) be a Lie algebra and \( V \) an ideal of \( G \). If there exists a subalgebra \( U \) of \( G \) such that \( G = U + V \) and \( U \cap V = (0) \) we say that \( G \) splits over \( V \).

As a corollary of 1.2 we have

1.7. Let \( G \) be a Lie algebra over a ground field of characteristic 0 and let \( L \supseteq M \) be ideals of \( G \) such that \( G/L \) is semi-simple, \( M \) is abelian and \( L \) splits over \( M \). Then \( G \) splits over \( M \).

It would be interesting to know if the hypothesis that \( M \) be abelian can be removed from 1.7.

2. Module extensions. Let \( P \) and \( Q \) be \( G \)-modules, \( L \) any ideal of \( G \). (We drop the requirement that \( G/L \) be semi-simple and that the ground field have characteristic 0.) If \( P \subseteq E \) and if the sequence \( 0 \rightarrow P \rightarrow E \rightarrow \phi Q \rightarrow 0 \) of \( L \)-modules is exact we say that the pair \((E, \phi)\) is an \( L \)-extension of \( P \) by \( Q \). Given two \( L \)-extensions \((E_1, \phi_1)\) and \((E_2, \phi_2)\) of \( P \) by \( Q \), they are called \( L \)-equivalent if there is an \( L \)-isomorphism \( \tau \) of \( E_1 \) onto \( E_2 \) such that \( \tau(\phi) = \phi \) for all \( \phi \) in \( P \) and such that \( \phi \circ \tau = \phi_1 \). We denote by \( \{(E, \phi)\} \) the class of extensions \( L \)-equivalent to \((E, \phi)\) and recall that these classes form a vector space, over the ground field of \( L \), which we denote by \( \text{Ext}_L [P, Q] \).

Let \( M \) be the vector space of linear maps of \( Q \) into \( P \) made into a \( G \) (and thus also an \( L \))-module by setting

\[
(\gamma \cdot f)(q) = \gamma \cdot f(q) - f(\gamma \cdot q) \quad \text{for all } \gamma \text{ in } G, f \text{ in } M
\]

and \( q \) in \( Q \). It is known that \( \text{Ext}_L [P, Q] \cong H^1(L, M) \) as a vector space \([2, p. 562]\) and we now define an operation of \( G \) on \( \text{Ext}_L [P, Q] \) which agrees with the operation of \( G \) on \( H^1(L, M) \) in the sense that this vector space isomorphism becomes a \( G \)-isomorphism. Take \( \gamma \) in \( G \), and \((E, \phi)\) an \( L \)-module extension of \( P \) by \( Q \). As a vector space, put

\[
E^* = \{(e_1, e_2) \mid e_1, e_2 \text{ in } E \text{ and } \gamma \cdot \phi(e_1) = \phi(e_2)\}.
\]
Make $E^*$ into an $L$-module such that

$$x \cdot (e_1, e_2) = (x \cdot e_1, x \cdot e_2 + [\gamma, x] \cdot e_1)$$

for $x$ in $L$ and $(e_1, e_2)$ in $E^*$. Define a map $\phi^*$ of $E^*_\gamma$ onto $Q$ by $\phi^*(e_1, e_2) = \phi(e_1)$. The kernel of $\phi^*_\gamma$ is $(P, P)$ and the map $w_\gamma: (P, P) \to P$ such that

$$w_\gamma(p_1, p_2) = \gamma \cdot p_1 - p_2$$

is an $L$-homomorphism of $(P, P)$ onto $P$. The kernel $N_\gamma$ of $w_\gamma$ consists of all pairs $(p_1, p_2)$ for which $p_2 = \gamma \cdot p_1$. $N_\gamma$ is an $L$-submodule of $E^*_\gamma$ so that we may use $w_\gamma$ to induce an $L$-identification of $(P, P)/N_\gamma$ with $P$. Finally we put

$$\gamma^* = (E^*_\gamma/N_\gamma, \phi^*_\gamma)$$

[$\phi^*$ is used again to denote the $L$-homomorphism of $E^*_\gamma/N_\gamma$ induced by the original $\phi^*$,]

Suppose now that $(E_1, \phi_1)$ is an extension of $P$ by $Q$ which is $L$-equivalent to $(E, \phi)$ and let $\psi$ be an $L$-isomorphism of $E$ onto $E_1$. The map $\tilde{\psi}: E^*_\gamma \to E^*_1$ such that

$$\tilde{\psi}(e_1, e_2) = (\psi(e_1), \psi(e_2))$$

is an $L$-homomorphism which takes $N_\gamma$ onto $N_{1,\gamma}$ and induces an $L$-isomorphism of $E^*_\gamma/N_\gamma$ onto $E^*_1/N_{1,\gamma}$. Thus we may put

$$\gamma\{ (E, \phi) \} = \{ \gamma^*(E, \phi) \}.$$  

2.1. Proposition. If $h$ is the element of $H^1(L, M)$ determined by $(E, \phi)$ then the element of $H^1(L, M)$ determined by $\gamma^*(E, \phi)$ is $\gamma \cdot h$ [$M = \text{linear maps of } Q \text{ into } P$].

Proof. Let $\psi$ be a linear map of $Q$ into $E$ such that $\phi_\psi(q) = q$ for all $q$ in $Q$. Then we know that the $f$ in $C^1(L, M)$ such that

$$f(x)\{ q \} = x \cdot \psi(q) - \psi(x \cdot q)$$

for all $x$ in $L$

is an element of $Z^1(L, M)$ which represents $h$. Define a linear map $\varphi^*$ of $Q$ into $E^*_\gamma/N_\gamma$ by

$$\varphi^*(q) = (\psi(q), \psi(\gamma \cdot q)) + N_\gamma.$$  

Then $\phi^*_\gamma \varphi^*(q) = q$ for all $q$ in $Q$ and

$$x \cdot \varphi^*(q) - \varphi^*(x \cdot q) = (\gamma \cdot f)(x)\{ q \}$$

for all $x$ in $L$.

2.2. Corollary. The operation of $G$ on $\text{Ext}_L[P, Q]$ induces the structure of a $G$-module on $\text{Ext}_L[P, Q]$ and the vector space isomorphism
of \( \Ext_L [P, Q] \) onto \( H^1(L, M) \) becomes a \( G \)-isomorphism.

Now define a map \( \rho: \Ext_G [P, Q] \to \Ext_L [P, Q] \) as follows. If \((E, \phi)\) is a \( G \)-extension of \( P \) by \( Q \), define an \( L \)-extension \((\overline{E}, \overline{\phi})\) of \( P \) by \( Q \) by restricting the operators on \( E \) to elements of \( L \). As an \( L \)-module, \( \overline{E} \equiv E \) and as an \( L \)-homomorphism \( \overline{\phi} \equiv \phi \). It is evident that if \( h \) is the element of \( H^1(G, M) \) determined by \( \{(E, \phi)\} \) then \( \rho_1(h) \) is the element of \( H^1(L, M) \) determined by \( \rho(\{(E, \phi)\}) \) where \( \rho_1 \) is the restriction homomorphism. Thus we have:

2.3. Proposition. \( \rho(\Ext_G [P, Q]) \subseteq \Ext_L [P, Q]^G. \)

Now we have the diagram:

\[
\begin{array}{ccc}
\Ext_G [P, Q] & \overset{\rho}{\longrightarrow} & \Ext_L [P, Q]^G \\
\downarrow & & \downarrow \\
H(G, M) & \overset{\rho_1}{\longrightarrow} & H^1(L, M)^G
\end{array}
\]

where \( \rho \) and \( \rho_1 \) are homomorphisms into. Further this diagram is commutative.

It is easy to find examples showing that \( \rho_1 \) (thus also \( \rho \)) need not be an isomorphism in general. Moreover, the following example (due to Hochschild) shows that \( \rho_1 \) need not even be onto in general: \( G \) is 3 dimensional, with basis \( x, y, z \) such that \([x, y] = z \) and \( z \) is central. Put \( L = \langle z \rangle \), \( M \) a trivial 1 dimensional \( G \)-module. Then every 1-cocycle for \( G \) in \( M \) must map \( L \) into \( \langle 0 \rangle \); but the map \( f \), where \( f(az) = a \), represents a nonzero element of \( H^1(L, M)^G \).

In fact it is an easy consequence of 1.1 and Hochschild-Serre's Theorem 6 [1, p. 596] that

2.4. Proposition. If the ground field of \( G \) has characteristic 0 then \( \rho_1: H^1(G, M) \to H^1(L, M)^G \) is an isomorphism for all \( G \)-modules \( M \) if and only if \( G/L \) is semi-simple.

3. Lie algebra extensions. Let \( L \) be an ideal of \( G \) and \( M \) a \( G \)-module which we consider as an abelian Lie algebra. An extension of \( M \) by \( L \) is a pair \((U, \pi)\) such that \( M \) is an ideal of \( U \) and such that the sequence \( 0 \to M \to U \to L \to 0 \) is exact. Two such extensions \((U_1, \pi_1)\) and \((U_2, \pi_2)\) are called equivalent if there is an isomorphism \( \sigma: U_1 \to U_2 \) such that \( \sigma(m) = m \) for all \( m \) in \( M \) and \( \pi_1 = \pi_2 \sigma \). Using a procedure analogous to the Baer product, one can define directly the structure of a vector space on the classes of extensions \( \{(U, \pi)\} \). We denote this vector space by \( \Ext(M, L) \). Further it is known, [2, p. 566], that this vector space is isomorphic to \( H^2(L, M) \).

We define an operation of \( G \) on \( \Ext(L, M) \) as follows: take \( \gamma \) in
G and \((U, \pi)\) an extension of \(M\) by \(L\). Put
\[ U_\gamma = \{ \langle u_1, u_2 \rangle \mid u_1, u_2 \text{ in } U, [\gamma, \pi(u_1)] = \pi(u_2) \} \]
as a vector space and make \(U_\gamma\) into a Lie algebra by setting
\[ \left[ \langle u_1, u_2 \rangle, \langle u_1', u_2' \rangle \right] = \left[ \langle u_1, u_2' \rangle, \left[ u_1, u_2 \right] + \left[ u_2, u_1' \right] \right] \]
for all \(\langle u_1, u_2 \rangle\) and \(\langle u_1', u_2' \rangle\) in \(U_\gamma\). Define a homomorphism \(\pi_\gamma\) of \(U_\gamma\) onto \(L\) by \(\pi_\gamma(u_1, u_2) = \pi(u_1)\). The kernel of \(\pi_\gamma\) is \(\langle M, M \rangle\). This situation makes \(\langle M, M \rangle\) into an \(L\)-module with \(L\) operating on \(\langle M, M \rangle\) via \(U_\gamma\). The map \(\omega_\gamma: \langle M, M \rangle \rightarrow M\) such that
\[ \omega_\gamma(m_1, m_2) = \gamma \cdot m_1 - m_2 \]
is an \(L\)-homomorphism of \(\langle M, M \rangle\) onto \(M\). The kernel \(M_\gamma\) of \(\omega_\gamma\) consists of pairs \(\langle m, \gamma \cdot m \rangle\) and \(M_\gamma\) is an ideal of \(U_\gamma\). Thus \(\pi_\gamma\) induces a homomorphism (still denoted by \(\pi_\gamma\)) of \(U_\gamma/M_\gamma\) onto \(L\) with kernel \(\langle M, M \rangle/M_\gamma\) which we identify with \(M\) using \(\omega_\gamma\). Now define
\[ \gamma \cdot \{ (U, \pi) \} = \{ \gamma \cdot (U, \pi) \} \]

If \((U_1, \pi_1)\) is equivalent to \((U, \pi)\) under the isomorphism \(\psi\), the map \(\psi^*\) such that \(\psi^* - \langle u_1, u_2 \rangle = \langle \psi(u_1), \psi(u_2) \rangle\) induces an equivalence isomorphism of \((U_\gamma/M_\gamma, \pi_\gamma)\) with \((U_1\gamma/M_1\gamma, \pi_1\gamma)\) so that we may put
\[ \gamma \cdot \{ (U, \pi) \} = \{ \gamma \cdot (U, \pi) \} \]

3.1. **Proposition.** Let \(h\) be the element of \(H^3(L, M)\) determined by \((U, \pi)\). Then the element of \(H^3(L, M)\) determined by \(\gamma \cdot \{ (U, \pi) \}\) is \(\gamma \cdot h\).

**Proof.** Let \(\alpha\) be a linear map of \(L\) into \(U\) such that \(\pi \alpha(x) = x\) for all \(x\) in \(L\). The element \(f\) of \(C^3(L, M)\) such that
\[ f(x_1, x_2) = \left[ \alpha(x_1), \alpha(x_2) \right] - \alpha([x_1, x_2]) \]
is in \(Z^3(L, M)\) and represents \(h\). Define a linear map \(\hat{\alpha}: L \rightarrow U_\gamma/M_\gamma\) by
\[ \hat{\alpha}(x) = \langle \alpha(x), \alpha([\gamma, x]) \rangle + M_\gamma. \]
Then
\[ [\hat{\alpha}(x_1), \hat{\alpha}(x_2)] - \hat{\alpha}[x_1, x_2] = (\gamma \cdot f)(x_1, x_2) \quad \text{for all } x_1, x_2 \text{ in } L. \]

3.2. **Corollary.** The operation of \(G\) on \(\text{Ext}(M, L)\) induces the structure of a \(G\)-module on \(\text{Ext}(M, L)\) and the vector space isomorphism of \(\text{Ext}(M, L)\) onto \(H^2(L, M)\) becomes a \(G\)-isomorphism.

Next consider \(\text{Ext}(M, G)\). If \(\{(U, \pi)\}\) is in \(\text{Ext}(M, G)\) put
\[ \rho \{ (U, \pi) \} = \{ \pi^{-1}(L, \pi) \} \]
in $\text{Ext}(M, L)$ where $\bar{\pi}$ is the restriction of $\pi$ to $\pi^{-1}(L)$. $\rho$ is a homomorphism of $\text{Ext}(M, G)$ into $\text{Ext}[M, L]$ and it is evident that the cohomology class in $H^2(L, M)$ belonging to $\rho\{(U, \pi)\}$ is $\rho_2(h)$ where $g$ is the cohomology class in $H^2(G, M)$ belonging to $\{(U, \pi)\}$ and $\rho_2$ is the restriction homomorphism.


Proof. $\bar{U} = \pi^{-1}(L)$ is an ideal in $U$. Let $\bar{\pi} = \pi\big|_{\bar{U}}$ then $\gamma \cdot (\bar{U}, \bar{\pi})$ is equivalent to the split extension $(L, M)$ of $M$ by $L$. In fact, the map

$$(\tilde{u}_1, \tilde{u}_2) + M_{\gamma} \rightarrow (\bar{\pi} (\tilde{u}_1), \tilde{u}_2 - [\gamma^1, \tilde{u}_1])$$

of $U/\gamma M$ onto $(L, M)$ is an equivalence isomorphism.

Again we have a commutative diagram:

$$
\begin{array}{ccc}
\text{Ext } (M, G) & \rightarrow & \text{Ext } (M, L)^G \\
\Uparrow & & \Uparrow \\
H^2(G, M) & \rightarrow & H^2(L, M)^G
\end{array}
$$

where $\rho$ and $\rho_2$ are homomorphisms into. If $G/L$ is semi-simple and if the ground field has characteristic 0 we know that $\rho_2$ is an isomorphism onto by 1.1. Thus

3.5. Proposition. If $G/L$ is semi-simple and if the ground field of $G$ has characteristic 0, $\rho$ is an isomorphism of $\text{Ext } (M, G)$ onto $\text{Ext } (M, L)^G$.

4. Lie algebra kernels. As before, let $G$ be a Lie algebra, $L$ an ideal of $G$ and $M$ a $G$-module. The derivations of $L$ form a Lie algebra (denoted by $D(L)$) and the inner derivations form an ideal of $D(L)$ denoted by $I(L)$. An $L$-kernel with nucleus $M$ is a pair $(K, \phi)$ consisting of a Lie algebra $K$ and a homomorphism $\phi$ of $L$ into $D(K)/I(K)$, $M$ is required to be the center of $K$ and $\phi$ must induce the given $L$-module structure on $M$.

Hochschild [3, p. 699 et seq.], has defined a rule of composition of $L$-kernels and a partitioning of the set of $L$-kernels with nucleus $M$ into similarity classes such that the resulting set of similarity classes of kernels, $\text{Kern } [L, M]$, becomes a vector space over the same ground field as $L$. It is not difficult to show that each element of $\text{Kern } [L, M]$ determines a unique element of $H^3(L, M)$ and that this correspondence is in fact a vector space isomorphism of $\text{Kern } [L, M]$ into $H^3(L, M)$. This isomorphism is not onto in general and again we refer to [3] and [4] for the details.
Here we shall define directly an operation of $G$ on $\text{Kern } [L, M]$ which is "correct" in the sense that if the similarity class $\{[K, \phi]\}$ determines $h$ in $H^3(L, M)$ then $\gamma \cdot \{[K, \phi]\}$ determines $\gamma \cdot h$. An immediate consequence of this (at least if the ground field has characteristic 0) is that the effaceable elements of $H^3(L, M)$ constitute a $G$-submodule of $H^3(L, M)$. See [4, Theorem 5, p. 777].

In order to do this, let $(K, \phi)$ be an $L$-kernel with nucleus $M$ where $L$ is an ideal of $G$ and take $\gamma$ in $G$. Form the Lie algebra $K_\gamma$ of pairs $\langle k_1, k_2 \rangle$ with $k_1, k_2$ in $K$ and put

$$\langle [k_1, k_2], \langle k_1', k_2' \rangle \rangle = \langle [k_1, k_1'], [k_1, k_2'] + [k_2, k_1'] \rangle.$$  

$K_\gamma$ has center $\langle M, M \rangle$ and we define a homomorphism $t_\gamma: \langle M, M \rangle \to M$ such that $t_\gamma \langle m_1, m_2 \rangle = \gamma \cdot m_1 + m_2$. The kernel $T_\gamma$ of $t_\gamma$ consists of pairs $\langle m, -\gamma \cdot m \rangle$ with $m$ in $M$.

Now form $K_\gamma / T_\gamma$. The center of $K_\gamma / T_\gamma$ is $\langle M, M \rangle / T_\gamma$; for if $\langle [a_1, a_2], \langle k_1, k_2 \rangle \rangle$ is in $T_\gamma$ for all $\langle k_1, k_2 \rangle$ in $K_\gamma$ then $[a_1, k_1]$ is in $M$ for all $k_1$ in $K$ and $[a_1, k_1] + [a_2, k_1] = -\gamma \cdot [a_1, k_1]$. In particular, for pairs of the form $\langle m, k \rangle$ with $m$ in $M$, we get

$$[a_1, k] + [a_2, m] = -\gamma \cdot [a_1, m] = 0$$

whence $[a_1, k] = 0$ for all $k$ in $K$ so that $a_1$ is in $M$. Thus since $[a_1, k_2] + [a_2, k_1] = -\gamma \cdot [a_1, k_1]$ for all $\langle k_1, k_2 \rangle$, we see that $[a_2, k_1] = 0$ for all $k_1$ in $K$ i.e. $a_2$ is in $M$.

We define a homomorphism $\phi_\gamma$ of $L$ into $D(K_\gamma / T_\gamma) / I(K_\gamma / T_\gamma)$ as follows: let $\rho$ be any linear map of $L$ into $D(K)$ which is compatible with $\phi$ (i.e. the class mod $I(K)$ of $\rho(x)$ is $\phi(x)$ for all $x$ in $L$.) Set

$$\rho^*(x)\langle \langle k_1, k_2 \rangle + T_\gamma \rangle = \langle \rho(x) \langle k_1 \rangle, \rho(x) \langle k_2 \rangle - \rho([\gamma, x]) \langle k_1 \rangle \rangle + T_\gamma.$$  

It is easy to check that $\rho^*$ is a linear map of $L$ into $D(K_\gamma / T_\gamma)$ and that the class mod $I(K_\gamma / T_\gamma)$ of $\rho^*(x)$ does not depend on the choice of $\rho$ compatible with $\phi$. Now we note that if

$$[\rho(x_1), \rho(x_2)] - \rho([x_1, x_2]) = D_{r^*(x_1, x_2)}$$  

in $I(K)$ for $x_1, x_2$ in $L$, then

$$[\rho^*(x_1), \rho^*(x_2)] - \rho^*([x_1, x_2]) = D_{\tau^*(x_1, x_2)}$$  

in $I(K_\gamma / T_\gamma)$ where $\tau^*(x_1, x_2) = \langle \tau(x_1, x_2), -\tau([\gamma, x_1], x_2), -\tau(x_1, [\gamma, x_2]) \rangle + T_\gamma$. Now put $\phi_\gamma(x) = \phi^*(x) + I(K_\gamma / T_\gamma)$. One sees immediately that $\phi_\gamma$ induces the given $L$-module structure on the center $\langle M, M \rangle / T_\gamma$ and that $t_\gamma$ induces an $L$-isomorphism of $\langle M, M \rangle / T_\gamma$ onto $M$ so that the pair $(K_\gamma / T_\gamma, \phi_\gamma)$ is an $L$-kernel with nucleus $M$ which we denote by $\gamma \cdot (K, \phi)$.  

The function \( \tau : L \times L \to K \) is called a hindrance of \((K, \phi)\) and its (formal) coboundary \( \delta \tau \) is an element of \( Z^2(L, M) \) called a primary deviation. The cohomology class of \( \delta \tau \) (called the obstruction of \((K, \phi)\)) is independent of the choice of \( \rho \) compatible with \( \phi \) and of the choice of \( \tau \) such that \( \rho([x_1, x_2]) = D_{\tau(x_1, x_2)} \).

In this terminology, \( \tau^* (x_1, x_2) \) is a hindrance of \( \gamma \cdot (K, \phi) \) and a straightforward computation shows that

\[
\delta \tau^* = \gamma \cdot \delta \tau.
\]

Since two \( L \)-kernels have the same obstruction if and only if they are similar, we have

4.1. Proposition. The map \( \{(K, \phi)\} \to \{\gamma \cdot (K, \phi)\} \) (where braces indicate similarity class) induces the structure of a \( G \)-module on \( \text{Kern } [L, M] \) and further, if \( h \) is the obstruction of \( \{(K, \phi)\} \), then \( \gamma \cdot h \) is the obstruction of \( \{\gamma \cdot (K, \phi)\} \).

As before we define a map \( \rho \) of \( \text{Kern } [G, M] \) into \( \text{Kern } [L, M] \) by setting \( \rho \{(\hat{K}, \hat{\phi})\} = \{(\hat{K}, \hat{\phi}|_L)\} \) where \( \hat{\phi}|_L \) is the restriction of \( \hat{\phi} \) to \( L \).

It is evident that if \( h \) is the obstruction of \( \{(\hat{K}, \hat{\phi})\} \) then \( \rho_3(h) \) is the obstruction of \( \{(\hat{K}, \hat{\phi}|_L)\} \) where \( \rho_3 \) is the restriction map of \( H^3(G, M) \) into \( H^3(L, M) \). Again we have a commutative diagram:

\[
\begin{array}{ccc}
H^3(G, M) & \to & H^3(L, M)^G \\
\downarrow & & \downarrow \\
\text{Kern } [G, M] & \to & \text{Kern } [L, M]^G
\end{array}
\]

Appendix

For details concerning the following discussion the reader is referred to [5].

A. Group enlargements.

Definition. Let \( G \) be an arbitrary group (written multiplicatively) and \( P, Q \) groups on which \( G \) operates from the left. In this section, \( P \) is assumed commutative but \( Q \) need not be. A \( G \)-enlargement of \( P \) by \( Q \) is a pair \((E, \phi)\) such that

(i) \( G \) operates on \( E \) from the left,

(ii) \( P \) is a subgroup of \( E \) and a direct summand of \( E \) (as a group),

(iii) \( \phi \) is a \( G \)-homomorphism of \( E \) onto \( Q \) with kernel \( P \).

Two \( G \)-enlargements \((E_1, \phi_1), (E_2, \phi_2)\) are called equivalent if there exists a \( G \)-isomorphism \( \tau : E_1 \to E_2 \) such that \( \tau(p) = p \) for all \( p \) in \( P \) and \( \phi_2 \tau = \phi_1 \).

Eilenberg gives a definition of multiplication of equivalence classes.
of $G$-enlargements so that these classes form an abelian group denoted by $\text{Enl}(G, Q, P)$. If $\theta$ is an element of $\text{Hom}(Q, P)$ and if one puts $(\sigma \cdot \theta) \{q\} = \sigma \cdot \theta(q^{-1}q)$ for all $\sigma$ in $G$ and $q$ in $Q$ then $\text{Hom}(Q, P)$ becomes a $G$-module and Eilenberg then showed that $\text{Enl}(G, Q, P) \cong H^1(G, \text{Hom}(Q, P))$.

Now suppose that $G$ is a normal subgroup of a group $\Gamma$ and that $\Gamma$ also operates on $P$ and $Q$. Let $(E, \phi)$ denote a $G$-enlargement of $P$ by $Q$. Take $\gamma$ in $\Gamma$ and let $E_\gamma$ be the subgroup of $E \times E$ consisting of pairs $(e_1, e_2)$ such that $\phi(e_1) = \gamma \cdot \phi(e_2)$. Make $E_\gamma$ into a $G$-module such that $g \cdot (e_1, e_2) = (g \cdot e_1, \gamma \cdot g^{-1}e_2)$ for all $g$ in $G$. Let $\phi^\gamma$ denote the $G$-homomorphism $(e_1, e_2) \mapsto \phi(e_1)$ of $E_\gamma$ onto $Q$. The kernel of $\phi^\gamma$ is $(P, P)$ and the map $\alpha_{\gamma} : (P, P) \to P$ such that $\alpha_{\gamma}(p_1, p_2) = \gamma \cdot p_2$ is a $G$-homomorphism of $(P, P)$ whose kernel ker $(\alpha_{\gamma})$ consists of pairs $(p, 1)$ where $1$ is the identity of $P$. $\phi^\gamma$ induces a $G$-homomorphism $\phi$ of $E_\gamma / \ker(\alpha_{\gamma})$ onto $Q$ so that the pair $(E_\gamma / \ker(\alpha_{\gamma}), \phi, \gamma)$ is a $G$-enlargement of $P$ by $Q$ which we denote by $\gamma \cdot (E, \phi)$. It is not difficult to show that if $(E, \phi)$ is equivalent to $(E', \phi')$ then $\gamma \cdot (E, \phi)$ is equivalent to $\gamma \cdot (E', \phi')$ and that this operation induces the structure of a $\Gamma$-module on $\text{Enl}(G, Q, P)$. Further one can check that this structure is compatible with the $\Gamma$-module structure on $H^1(G, \text{Hom}(Q, P))$ so that the natural isomorphism of $\text{Enl}(G, Q, P)$ onto $H^1(G, \text{Hom}(Q, P))$ becomes a $\Gamma$-isomorphism.

B. Group extensions. If $M$ is a subgroup of $U$ and if the sequence $0 \to M \to U \to \pi G \to 0$ is exact then we say that the pair $(U, \pi)$ is an extension of $G$ by $M$. If $M$ is abelian, this situation makes $M$ into a $G$-module. Two such extensions $(U_1, \pi_1)$ and $(U_2, \pi_2)$ of $G$ by $M$ are called equivalent if there exists an isomorphism $\tau : U_1 \to U_2$ such that $\tau(m) = m$ for all $m$ in $M$ and such that $\pi_1 = \pi_2 \tau$. The equivalence classes of such extensions form an abelian group $A(G, M)$ which is isomorphic with $H^2(G, M)$ see [5].

Now suppose that $G$ is a normal subgroup of a group $\Gamma$ and that $M$ is also a $\Gamma$-module. Let $(U, \pi)$ be an extension of $G$ by $M$ and take $\gamma$ in $\Gamma$. Form $U_\gamma = \{ \langle u_1, u_2 \rangle : u_1$ in $U, \pi(u_1) = \pi(u_2)\gamma \}$ with $\langle u_1, u_2 \rangle \langle u_1', u_2' \rangle = \langle u_1 u_1', u_2 u_2' \rangle$. Set $\pi_\gamma(u_1, u_2) = \pi(u_1)$ and note that the kernel of $\pi_\gamma$ is $\langle M, M \rangle$. Define a homomorphism $\omega_\gamma : \langle M, M \rangle \to M$ by $\omega_\gamma(m_1, m_2) = \gamma \cdot m_2$ for all $\langle m_1, m_2 \rangle$ in $M$. $\omega_\gamma$ is a $G$-homomorphism of $\langle M, M \rangle$ onto $M$ and the kernel of $\omega_\gamma = \langle M, 0 \rangle$ is a normal subgroup of $U_\gamma \cdot \pi_\gamma$ induces a homomorphism (still denoted by $\pi_\gamma$) of $U_\gamma / \langle M, 0 \rangle$ onto $M$ and we put

$$\gamma \cdot (U, \pi) = (U_\gamma / \langle M, 0 \rangle, \pi_\gamma).$$

One can check that if $(U, \pi)$ is equivalent to $(U_1, \pi_1)$ then $\gamma \cdot (U, \pi)$
is equivalent to $\gamma \cdot (U, \pi)$ and that this operation induces the structure of a $\Gamma$-module on $A(G, M)$. Further, if $h$ is the element of $H^3(G, M)$ belonging to $(U, \pi)$ then $\gamma \cdot h$ belongs to $\gamma \cdot (U, \pi)$ so that the isomorphism of $A(G, M)$ onto $H^3(G, M)$ becomes a $\Gamma$-isomorphism.

C. **Group kernels.** A $G$-kernel is a pair $(K, \theta)$ where $\theta$ is a homomorphism: $G \rightarrow A(K)/I(K)$ (the outer automorphisms of $K$). If $M$ is the center of $K$, then $\theta$ makes $M$ into a $G$-module called the nucleus of $(K, \theta)$. Eilenberg-MacLane [5, p. 12] have given a rule of composition of $G$-kernels with fixed nucleus $M$ and a partitioning of the set of $G$-kernels with nucleus $M$ into similarity classes such that the resulting set of similarity classes $\mathcal{K}(G, M)$ becomes an abelian group. They then show that $\mathcal{K}(G, M)$ is isomorphic to $H^3(G, M)$.

Now suppose that $G$ is a normal subgroup of a group $\Gamma$ and that $M$ is also a $\Gamma$-module. Let $(K, \theta)$ be a $G$-kernel with nucleus $M$ and take $\gamma$ in $\Gamma$. As a group, put $K_\gamma = K$ and put $\theta_\gamma(\sigma) = \theta(\gamma^{-1}\sigma\gamma)$ for all $\sigma$ in $G$. The center $M_\gamma$ of $K_\gamma$ is still $M$ (as a group) but now the operation of $G$ on $M_\gamma$ is given by

$$\sigma \ast m = \gamma^{-1}\sigma\gamma \cdot m$$

for all $\sigma$ in $G$, $m$ in $M$.

The map $m \mapsto \gamma \cdot m$ is a $G$-isomorphism of $M_\gamma$ onto $M$ which we use to identify $M_\gamma$ with $M$ as a $G$-module. Then the pair $(K_\gamma, \theta_\gamma)$ is a $G$-kernel with nucleus $M$ which we denote by $\gamma \cdot (K, \theta)$. One can check that this operation induces the structure of a $\Gamma$-module on $\mathcal{K}(G, M)$ and sees that if $h$ is the element of $H^3(G, M)$ belonging to $(K, \theta)$ then $\gamma \cdot h$ belongs to $\gamma \cdot (K, \theta)$ so that the isomorphism of $\mathcal{K}(G, M)$ with $H^3(G, M)$ becomes a $\Gamma$-isomorphism.

**Bibliography**


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