

NONCOUNTABLE NORMALLY LOCALLY FINITE DIVISION ALGEBRAS

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A (commutative) field F is *regular* (see [1] of the bibliography) if it is not finite, and if in addition it is true that the direct (= Kronecker = tensor) product of two normal (= central) division algebras, of finite orders, over F is not a division algebra unless their orders are relatively prime; algebraic number fields and p -adic fields are examples of regular fields. A division algebra \mathfrak{A} over a field F is *normally locally finite* if any finite subset of \mathfrak{A} is contained in a normal (over F) division sub-algebra of \mathfrak{A} of finite order; in [1], such algebras were called "of type 1." A *subisomorphism* of an algebra \mathfrak{A} over F is an algebra-isomorphism of \mathfrak{A} into \mathfrak{A} , and it is *proper* if it is not onto. If \mathfrak{A} is a normally locally finite division algebra over the regular field F , without a finite basis over F , a *characteristic sub-algebra* of \mathfrak{A} is any normally locally finite division sub-algebra \mathfrak{D} of \mathfrak{A} , with countably infinite basis over F , having the property that any normally locally finite division sub-algebra of \mathfrak{A} , with finite or countable basis over F , is isomorphic to a sub-algebra of \mathfrak{D} . It was proved in [1] that any \mathfrak{A} of the previous type has a characteristic sub-algebra, unique but for isomorphisms; it was also proved that there exists a normally locally finite division algebra over the regular field F , with infinite noncountable basis, and with a given characteristic sub-algebra \mathfrak{D} , if and only if \mathfrak{D} admits proper subisomorphisms; [1] contains a rather involved proof of the fact that any \mathfrak{D} admits proper subisomorphisms if F is not countable, and thus establishes the existence of normally locally finite division algebras, with infinite noncountable basis, over any noncountable regular field; this seems to be the only known example of such algebras. We shall present here a very simple proof of the same result, and will, at the same time, dispense with the condition of noncountability of F .

(1). LEMMA. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ be normal division algebras, of finite orders > 1 , over the (certainly infinite) field F , and suppose $\mathfrak{A} \times \mathfrak{B} \times \mathfrak{C}$ also to be a division algebra; let m be an element of $\mathfrak{A} \times \mathfrak{B}$ but not of \mathfrak{A} . Then there exists a $d \in \mathfrak{B} \times \mathfrak{C}$, not zero, such that $d^{-1}md \notin \mathfrak{A} \times \mathfrak{B}$.*

In the previous statement, as in the rest of this paper, the identity

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elements of the direct factors of a direct product of algebras are assumed to be coincident.

PROOF (being a modification, due to D. Zelinsky, of a proof by the author). Let c be an element of \mathbb{C} , but not of F , and let b be an element of \mathfrak{B} such that $mb \neq bm$; such b exists because the commutator (=centralizer) of \mathfrak{B} in $\mathfrak{A} \times \mathfrak{B}$ is \mathfrak{A} , and $m \notin \mathfrak{A}$. Set $x = bc \neq 0$, so that also $1 + x \neq 0$; if the lemma is false, we have $xy = mx$, $(1 + x)z = m(1 + x)$ for suitable elements y, z of $\mathfrak{A} \times \mathfrak{B}$; subtracting, we obtain $m = z + x(z - y)$. If $y = z$, then $m = z = y$, $xm = mx$, $bm = mb$, a contradiction; if $y \neq z$, then $bc = x = (m - z)(z - y)^{-1} \in \mathfrak{A} \times \mathfrak{B}$, also a contradiction, since $c \notin F$, Q.E.D.

For the convenience of the reader, we repeat here a portion of the statement of (6) of [1]:

(2). LEMMA. *Let \mathfrak{A} be a normally locally finite division algebra, with countably infinite basis, over the field F ; a necessary and sufficient condition in order that \mathfrak{A} admit a proper subisomorphism is that there exist a factorization*

$$\mathfrak{A} = \mathfrak{B}_0 \times \mathfrak{B}_1 \times \dots$$

of \mathfrak{A} as a direct product of normal division algebras of finite orders > 1 over F , an $m \in \mathfrak{B}_0$, and a sequence h_1, h_2, \dots of elements of \mathfrak{A} , such that, after setting $\mathfrak{A}_i = \mathfrak{B}_0 \times \mathfrak{B}_1 \times \dots \times \mathfrak{B}_i$, the following conditions be satisfied:

- (a) $h_i \in \mathfrak{A}_i$;
- (b) $h_{i+1} = h_i c_i$ for a $c_i \in \mathfrak{B}_i \times \mathfrak{B}_{i+1}$;
- (c) there exists no $z_{i-1} \in \mathfrak{A}_{i-1}$ such that $h_i z_{i-1} = m h_i$.

We can now prove:

(3). THEOREM. *Let \mathfrak{A} be as in (2); then \mathfrak{A} admits a proper subisomorphism.*

PROOF. By (29) of [1],¹ \mathfrak{A}_0 cannot be transformed into itself by every inner automorphism of \mathfrak{A}_1 ; hence there exist an $m \in \mathfrak{A}_0$, and an $h_1 \in \mathfrak{A}_1$, with $h_1 \neq 0$, such that $h_1^{-1} m h_1 \notin \mathfrak{A}_0$. We shall now proceed to build the sequence $\{h_i\}$ of (2) by induction: assume the h_1, \dots, h_i to have been found; by (1) (after replacing \mathfrak{A} by \mathfrak{A}_{i-1} , \mathfrak{B} by \mathfrak{B}_i , \mathbb{C} by \mathfrak{B}_{i+1} , m by $h_i^{-1} m h_i$), there exists a $c_i \in \mathfrak{B}_i \times \mathfrak{B}_{i+1}$, not zero, such that $c_i^{-1} (h_i^{-1} m h_i) c_i \in \mathfrak{A}_i$. Then $h_{i+1} = h_i c_i$ satisfies the conditions of (2), Q.E.D.

¹ This is the little theorem with a distinguished career, first proved in [2], which later came to be known as the Cartan-Brauer-Hua theorem (see for instance [3, Chapter VII, §13]); the proof given in [1] is the first elementary proof for the finite case.

(4). COROLLARY. *Let \mathfrak{A} be as in (2), and assume F to be regular; then there exists a normally locally finite division algebra over F , with infinite noncountable basis, having \mathfrak{A} as characteristic sub-algebra.*

On the other hand, if F is not regular, the concept of characteristic sub-algebra loses meaning; however, from (3), and from a slight modification of the construction used to prove the sufficiency of (3) of [1], we still obtain:

(5). COROLLARY. *Let F be a field such that there exists a normally locally finite division algebra \mathfrak{A} with countably infinite basis over F ; then there exists a normally locally finite division algebra over F , with infinite noncountable basis, having \mathfrak{A} as a sub-algebra.*

REMARK. An examination of the proof of (3) of [1] discloses that all the normally locally finite division algebras, with infinite noncountable basis over F , whose existence has been established in this note, have a basis of cardinality \aleph_1 ; the existence of normally locally finite division algebras over F , with a basis of cardinality $> \aleph_1$, is still an open problem, at least when F is regular.

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