**D-REGULARITY**

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We shall call an element \( x \) of a ring \( A \), right \( D \)-regular if there exists an element \( y \) in \( A \) such that \( x = xy \). This property of \( x \) belonging to \( xA \) has been studied before [2; 6].\(^1\) With techniques available from [5] it is not difficult to show the existence of a maximal right \( D \)-regular two-sided ideal \( M_R \) and a left analogue \( M_L \). These are in general not equal. They are connected with the Jacobson radical \( J \) and the subradicals \( P_R \) and \( P_L \), [6], in the following way:

\[
P_R = J \cap M_R; \quad P_L = J \cap M_L.
\]

The present note goes on to consider the cases \( M_R = 0 \) and \( M_R = A \), for various degrees of chain assumptions. In the commutative case \( M_R = M_L = M \), the maximal \( D \)-regular ideal.

1. **Preliminaries.** Following Brown and McCoy [5], to each element \( a \) of a ring \( A \) we associate the right ideal \( F(a) = aA \). The element \( a \) is said to be right \( D \)-regular (r.D.r.) if \( a \) belongs to \( F(a) \). A right ideal is said to be r.D.r. if every element in it is r.D.r. It is easy to see that \( F(a + b) \subseteq F(a) + (b) \subseteq F(a) + (b) \), and that \( F(a + b) \subseteq F(a) \) if \( b \) is in \( F(a) \). Then by Theorems 1 and 2 of [5] we can conclude that \( M_R = \{ x: (x) \) is r.D.r. \}, is a two-sided ideal which contains every r.D.r. two-sided ideal; and that \( M_R(A - M_R) = 0 \). In other words, \( A \) is an \((F, \Omega, \Omega')\) group and by Theorem 6 of [5] we have:

\[
M_R = \bigcap_i M_i',
\]

where \( M_i \) is a large modular right ideal, i.e. there exists an element \( x_i \) not in \( M_i \) such that \( x_iA \leq M_i \) and such that every right ideal which properly contains \( M_i \), also contains \( x_i \). The set \( M_i' \) is the largest two-sided ideal contained in \( M_i \).

Though this development is both elegant and general it does not seem to yield the fact that \( M_R \) contains all the r.D.r. right ideals. In particular, if \( x \) is in \( xA \) and if for every \( y \) of \( A \), \( xy \) is in \( xyA \), it is not clear that \( x \) must be in \( M_R \). To obtain this fact one must return to the original Jacobson techniques and develop \( M_R \) from a one sided point of view. Using a technique of [6] we obtain:

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\(^1\) Numbers in square brackets refer to the bibliography at the end of the paper.

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Lemma 1. If $x$ is r.D.r. and if $a$ belongs to an r.D.r. right ideal, namely $aA$, then $a + x$ is r.D.r.

Proof. We have $xx' = x$, $aa' = a$. Define $u = a - ax' = aa' - ax' = a(a' - x')$. Then $u$ belongs to $aA$ and therefore there exists an element $u'$ such that $uu' = u$. Define $v = u' + x' - x'u'$. Then $xv = xu' + xx'u' = xu' + x - xx'u' = x$; and $av = au' + ax' - ax'u' = (a - ax')u' + ax' = a - ax' + ax' = a$. Therefore $(a + x)v = a + x$. Q.E.D.

Corollary. The sum or two r.D.r. right ideals is an r.D.r. right ideal.

We define $M_R^*$ to be the union of all the r.D.r. right ideals. Then $M_R^*$ is itself an r.D.r. right ideal. We now show that it is a two-sided ideal:

Lemma 2. $M_R^*$ is a two-sided ideal of $A$.

Proof. For $a$ in $A$ and $x$ in $M_R^*$, $xa$ is in $M_R^*$. Since $xx' = x$ and for every $y$ of $A$, there is an element $y'$ such that $xy = xyy'$; $ax' = ax$ and $axyy' = axy$. Therefore $ax$ is in $M_R^*$. Q.E.D.

It is now easy to see that $M_R^* = M_R$ and we have

Theorem 1. If $x$ is in $xA$ and for every $y$ of $A$, $xy$ is in $xyA$, then $x$ is in $M_R$.

The proof of the next theorem is immediate from results in [6].

Theorem 2. If $J$ is the Jacobson radical of $A$, then $P_R = J \cap M_R = J(M_R)$; and $P_L = J \cap M_L = J(M_L)$.

Thus in the commutative case, the subradical is the radical of the max.D.r. ideal.

We shall now obtain some simple properties of $M_R$. First we observe that $M_R$ and $M_L$ may not be equal. The following example is due to Hopkins [8]. Let $A$ be the set of all $me + nu$, where $e^2 = e$, $u^2 = 0$, $eu = u$, $ue = 0$, and where $m$, $n$ are in a field $F$. Then $M_L = A$ whereas $M_R = 0$. The proof of the following lemma is immediate.

Lemma 3. If $B$ is an ideal of $A$, then $M_R(B) < B \cap M_R$.

The point is that they may not be equal. Let $B = J$, and let $A$ have a right unity element. Then $M_R(B) = 0$, whereas $B \cap M_R = J \cap A = J$.

We observe also that $M_R = M_RA^n$ for every $n$; $M_L = A^mM_L$, for every $m$.

Lemma 4. $M_R(M_R) \subseteq M_R^n$ for every $n$; $M_R(M_R) = M_R(M_R^n)$ for every $n$.

Proof. If $x$ is in $M_R(M_R)$, then in particular $x = xy$, with $y$ in $M_R$. 


Then \( x = xy^{n-1} \), which is in \( M^n_R \). The second half is also immediate, 
\( x = xy^n \), with \( y^n \) in \( M^n_R \).

Thus we have \( M_R > M^2_R > \cdots > M^n_R > \cdots > M_R(M_R) = M_R(M^n_R) = \cdots \).

This chain can in fact descend.

**Example 1.** Let \( A \) be the set of all finite sums \( \sum_i \alpha_i x^i + \sum_j \beta_j y^j \), where \( \alpha_i \) and \( \beta_j \) are in a field \( F \) and where \( x \) and \( y \) are indeterminates such that \( x' y = y' x = x \) for every \( i \) and \( j \). Then \( M = (x) \), \( M^2 = (x^2) \), \( \cdots \), \( M^n = (x^n) \), \( M(M) = \cdots = M(M^n) = \cdots = 0 \).

By methods almost the same as in [4], one can prove

**Theorem 3.** If \( A_n \) is the complete matric ring of order \( n \) over \( A \) then \( M_R(A_n) = (M_R(A))_n \); \( M_L(A_n) = (M_L(A))_n \).

This leads to

**Lemma 5.** If \( a_1, \cdots, a_n \) is any finite set of elements in \( M_R \), \( a_i = a_i a_i^t \), then there exists an element \( b \) in \( A \) but not necessarily in \( M_R \) such that \( a_i = a_i b \) for all the \( a_i \).

**Proof.** Since the \( a_i \) are in \( M_R \), the \( n \times n \) matrix \( c \), with the \( a_i \) in the first column and zeros elsewhere, is in \( M_R(A_n) \), by Theorem 3. Then, in particular, there exists a matrix \( d = (b_{ij}) \) such that \( cd = c \). However

\[
cd = \begin{pmatrix}
a_1 b_{11} & \cdots \\
a_2 b_{11} & \cdots \\
\vdots \\
a_n b_{11} & \cdots
\end{pmatrix}
\]

and therefore \( b = b_{11} \).

An inductive proof can also be given. For \( n = 2 \), Lemma 5 is a consequence of the proof of Lemma 1. We assume the result for any set of \( n-1 \) elements of \( M_R \). Given \( a_1, \cdots, a_n \), with \( a_i = a_i a_i^t \), consider the set of \( n-1 \) elements \( a_i - a_i a_n^t \), for \( i = 1, \cdots, n-1 \). By induction there exists an element \( g \) such that \( (a_i - a_i a_n^t) g = a_i - a_i a_n^t \). Define \( b = a_n^t + g - a_n^t g \). Then \( a_n b = a_n + a_n g - a_n g = a_n \); whereas \( a_i b = a_i a_i^t + (a_i - a_i a_n^t) g = a_i a_i^t + a_i - a_i a_n^t = a_i \), for \( i = 1, \cdots, n-1 \). Q.E.D.

**Corollary.** If \( M_R \) is finitely generated as a left \( A \)-module, then there exists an element \( e \) in \( A \) such that \( M_R = M_R e \) pointwise. The element \( e \) is not necessarily in \( M_R \), or an idempotent or unique.

It is clear that if \( A \) has a right unity element then \( A = M_R \). If
$A = J$, then both $M_R$ and $M_L$ are zero, since $x$ in $xJ$ or $Jx$ implies $x = 0$, [3]. Thus we might expect that if $M_R = A$ then $A$ is well behaved whereas if $M_R = M_L = 0$ then $A$ is radical-like. This is certainly so when $A$ is commutative with DCC. In that case $A$ can be expressed as $A = eA + N_0$ where $e$ is an idempotent, $eN_0 = 0$ and $N_0$ is nilpotent. Since a central idempotent is always in both $M_R$ and $M_L$, here $e$ is in $M$ and thus $eA \leq M$. On the other hand if $x$ is in $M$, $x = ex_1 + n_1$, then in particular there exists an element $ey + n_2$ such that $(ex_1 + n_1) \cdot (ey + n_2) = ex_1 + n_1$ and thus $n_1n_2 = n_1, n_1 = 0$. Therefore $M \leq eA, M = eA$. Also $N_0$ is simply $M'$, the set of annihilators of $M$. We have

**Theorem 4.** If $A$ is a commutative ring with DCC then $A = M + M'$, where $M$, the max.D.r. ideal, has a unity element and $M'$ is nilpotent.

This corresponds to the result in [4] which states that every ring $A$ with DCC (though not necessarily commutative) can be expressed as $M + M'$ where $M$, the max. regular ideal, is semi-simple, and where $M'$ is bound to its radical. Here $M > \bar{M}$ and $M' < \bar{M}$. Thus we know more about $M'$, namely that it is nilpotent, and less about $M$, since it is not necessarily semi-simple.

**Corollary 1.** If $A$ is commutative with DCC then $M = A$ if and only if $A$ has a unity element; $M = 0$ if and only if $A$ is nilpotent.

Using the fact that $M(A - M) = 0$ we then have

**Corollary 2.** If $A$ is commutative with DCC for $A - M$ or in particular for $A$, then $A - M$ is nilpotent.

Theorem 4 and its corollaries remain true if the condition of commutativity is relaxed to the restriction that all idempotents lie in the center. Without DCC however, Theorem 4 is false, for in Ex. 1, $M$ is the set of all $\sum_{i}^{n} x_i$, whereas $M'$ is the set of all $\sum_{j}^{m} y_j$ with $\sum_{i}^{n} \beta_i = 0$. Thus $M$ and $M'$ do not fill out all of $A$.

2. **The cases $M_R, M_L, M_R$ and $M_L$ equal to zero.**

**Theorem 5.** If $A$ is a ring with DCC on right ideals, then $A$ is nilpotent if and only if $M_R = 0$ and there are no nonzero absolute left zero divisors (i.e. elements $x$ such that $xA = 0$) in $A^n$ for every $n$.

**Proof.** In one direction the proof is clear. Assume then that $M_R = 0$ and that there are no nonzero absolute zero divisors in $A^n$. By DCC we can write $A = e_1A + \cdots + e_nA + N_0$ where the $e_iA$ are indecom-
posable right ideals and $N_0$ is nilpotent. If $e_1 \neq 0$ then it is not in $M_R$. Then there must exist an $x'$ in $A$ such that $e_1 x' \neq 0$ and such that $e_1 x'$ is not in $e_1 x'A$. Else $e_1$ is in $M_R$ by Theorem 1. Thus $e_1 x'A \neq e_1 A$. Since $e_1 A$ is indecomposable, $e_1 x'A$ must be nilpotent. Since $e_1 x' = e_1^{n-1} x'$ is in $A^n$ for every $n$, $e_1 x'A$ cannot be zero. Let $N$ be the max. nilpotent ideal of $A$. Then $e_1 N > e_1 e_1 x'A = e_1 x'A \neq 0$. However the chain $e_1 N > e_1 N^2 > \cdots > e_1 N^n > \cdots$ terminates in zero after a finite number of steps. Then there exists an integer $w \geq 1$ such that $e_1 N^w \neq 0$, $e_1 N^{w+1} = 0$. Let $x''$ be an element of $N^w$ such that $e_1 x'' \neq 0$ and let $x = e_1 x''$. Then $xN = 0$, $x \neq 0$. Since $xN < xN = 0$, there must exist an $e_i$ such that $xe_i A \neq 0$. However since $xN = 0$, $xe_i A$ is a minimal right ideal of $A$. For if $0 \neq I \subseteq xe_i A$, where $I$ is a right ideal of $A$, let $Q = \{y \in e_i A : xy \text{ is in } I\}$. Then $Q$ is a right ideal of $A$, $Q \subseteq e_i A$. If $Q \neq e_i A$, then $Q$ is nilpotent because $e_i A$ is indecomposable. Then $xQ < xN = 0$ and therefore if $z$ is in $I$, $z = xe_i z'$ for some $z'$ in $A$, $e_i z'$ is in $Q$, $xe_i z' = 0$, $I = 0$. Thus $Q = e_i A$, $I = xe_i A$.

Finally let $y'$ be an element of $A$ such that $xe_i y' \neq 0$, and let $y = e_i y'$. Then $xy$ is in $xe_i A$, $xyN = 0$, $xyA \neq 0$ (since $xy = e_i^{n-2} xy$ is in $A^n$ for every $n$). Now $xyA \subseteq xe_i A$ and since $xe_i A$ is minimal, $xyA = xe_i A$. Therefore $xy$ is in $xyA$. Furthermore if $xyu \neq 0$, $xyuA \neq 0$ (again since $xyu = e_i^{n-2} xyu$ is in $A^n$ for every $n$) and thus $xyu$ is in $xyuA$. Therefore $xy$ is in $M_R$ by Theorem 1. This is a contradiction and thus all the $e_i$ are zero, and $A$ must be nilpotent. Q.E.D.

**Corollary 1.** If $A$ has DCC on left ideals then $A$ is nilpotent if and only if $M_L = 0$ and there are no nonzero absolute right zerodivisors in $A^n$ for every $n$.

Since $M_R$ contains the max. regular ideal $M$ and when $A - J$ is regular, or in particular when $A$ has DCC on right ideals, then $M = 0$ if and only if $A$ is bound to $J$, Theorem 6, [4], we can conclude that when $M_R = 0$ and $A$ has DCC on right ideals, $A$ is bound to $J$. Combining this with Theorem 5 we have

**Corollary 2.** If $A$ has DCC on right ideals and $M_R = 0$, then either $A$ is nilpotent or $A$ is bound to $N$ and $A$ has an absolute left zero divisor in $A^n$ for every $n$.

The converse is also true. Using the fact that $M_R (A - M_R) = 0$ we have

**Corollary 3.** If $A$ has DCC on right ideals, then $A - M_R$ is nilpotent if and only if $xA \leq M_R$ and $x$ in $(A - M_R)^n$ for every $n$, implies that $x$ is in $M_R$. 

When \( M_R = M_L = 0 \) the zero divisor condition can be slightly weakened.

**Theorem 6.** If \( A \) has DCC on one-sided ideals, then \( A \) is nilpotent if and only if \( M_R = M_L = 0 \) and there are no nonzero total divisors of zero in \( A^n \) for every \( n \), i.e. elements \( x \) such that \( xA = Ax = 0 \) and \( x \) in \( A^n \) for every \( n \).

**Proof.** In one direction the proof is clear. Conversely, we write as before \( A = e_1A + \cdots + e_nA + N_0 \) where the \( e_iA \) are indecomposable right ideals and \( N_0 \) is nilpotent. If \( e_i \neq 0 \), consider the chain \( e_1N > e_1N^2 > \cdots > e_1N^\gamma = 0 \). There exists an integer \( \gamma \) such that \( e_1N^\gamma \neq 0 \), \( e_1N^\gamma+1 = 0 \). If \( \gamma = 0 \), let \( x \) be any element such that \( e_1x \neq 0 \). Consider \( e_1xA \leq e_1A \). If \( e_1xA \neq e_1A \), then since \( e_1A \) is indecomposable, \( e_1xA \) is nilpotent, \( e_1xA \) is in \( N \). Then \( e_1 \cdot e_1x = e_1N = 0 \). Then \( e_1x \) is properly nilpotent, \( e_1x \) is in \( N \). Then \( e_1 \cdot e_1x = 0 = e_1x \), a contradiction. On the other hand if \( e_1xA = e_1A \) then \( e_1x \) is in \( e_1xA \) and since \( e_1A \) is in \( A^n \), \( e_1A \) is in \( M_R \) by Theorem 1. Then \( e_1 = 0 \), a contradiction. Thus \( e_1N \neq 0 \), \( \gamma \geq 1 \). Similarly there exists an integer \( \rho \geq 1 \) such that \( N_0^\rho e_i \neq 0 \), \( N_0^{\rho+1}e_i = 0 \). In this way we obtain for each \( e_i \), integers \( \gamma_i \) and \( \rho_i \) such that \( e_iN_{\theta_i} \neq 0 \), \( e_iN_{\theta_i+1} = 0 \), \( N_{\theta_i}e_i \neq 0 \), \( N_{\theta_i+1}e_i = 0 \). Let \( \beta \) be the maximum of the \( \gamma_i \) and \( \rho_i \). Then setting \( e = e_1 + \cdots + e_n \), \( eN_{\beta+1} = N^{\beta+1}e = 0 \), and either \( eN_{\beta} \) or \( N_{\beta}e \neq 0 \). Suppose \( eN_{\beta} \neq 0 \). Then for some \( e_j, e_jN_{\beta} \neq 0 \). Take \( x' \) in \( N_{\beta} \) such that \( e_jx' \neq 0 \) and let \( x = e_jx' = e_je_jx' = e_jx = e_jx' \). Then \( xN = 0 \). Since \( M_L = 0 \) and \( x \) is in \( Ax \), there must exist (Theorem 1) an element \( y \) in \( A \) such that \( xy \neq 0 \) and such that \( xy \) is not in \( Axy \). We may take \( y = ye_j = ye_jx \). The element \( y \) must be in \( N \). For \( xy \) not in \( Ayx \) implies \( y \) not in \( Ay = Ay_ej \). Thus \( Ay_ej \neq Ae_j \) and since \( Ae_j \) is indecomposable, \( Aye_j \) is nilpotent. Then \( ye_j \cdot ye_j \) is nilpotent, \( ye_j \) is nilpotent and clearly \( ye_j = y \) is properly nilpotent and therefore \( y \) is in \( N \). Then \( xy \) is in \( Ne_jN_{\beta} \leq N_{\beta+1} \) and therefore \( xyx = 0 \). Also \( yxN = 0 \) and thus \( yxA = 0 \). Also \( eyx = 0 \), since \( eN_{\beta+1} = 0 \). Note that \( yx = ye_jx = ye_j^{\beta-2}x \) is in \( A^n \) for every \( n \). If \( xy \) is not a total zerodivisor, \( Axy \neq 0 \). Then, \( Nxy \neq 0 \), since \( eyx = 0 \). Let \( y_1 \) be an element in \( N \) such that \( y_1yx \neq 0 \). As above \( y_1yx = ey_1yx = 0 \). If \( y_1yx \) is not a total divisor of zero, \( Ny_1yx \neq 0 \). We continue this process until \( t = y_{\beta-1} \cdots y_1yx \neq 0 \), \( tA = 0 \), \( et = 0 \). Then \( t \) is in \( N_{\beta}e_jN_{\beta} \) and \( Nt \) is in \( N_{\beta+1}e_jN_{\beta} \). Thus \( At = tA = 0 \), and \( t = y_{\beta-1} \cdots y_1ye_j^n \) is in \( A^n \) for every \( n \). This is impossible and thus \( e_i = 0 \) for every \( i \), \( A = N_0 \), \( A \) is nilpotent. Q.E.D.

The Hopkins example mentioned earlier shows that the divisor of zero restrictions cannot be removed, for \( M_R = 0 \), \( A \) has DCC and is not nilpotent. To obtain an example for Theorem 6, let \( A \) be an algebra of dimension 4 over a field \( F \), with basal elements \( e, u, v, w \) and the following multiplication table:
Then $M_R = M_L = 0$ and $A$ is not nilpotent. The radical is generated by $u$, $v$, and $w$. The element $w$ is a total divisor of zero and in $A^n$ for every $n$. This algebra is in fact subdirectly irreducible with minimal ideal generated by $u$, $v$, and $w$. The element $w$ is a total divisor of zero and in $A^n$ for every $n$. This algebra is in fact subdirectly irreducible with minimal ideal generated by $w$. For let $I$ be any ideal of $A$, with $x = \alpha e + \beta u + \gamma v + \delta w$ in $I$. Then $xe = \alpha e + \beta u$, $ex = \alpha e + \gamma v$, $xv = \alpha v$, $ux = \alpha u$. Then if $\alpha \neq 0$, $I$ contains $u$, $v$ and then $w$, and also $e$, $I = A$. If $\alpha = 0$, $I$ contains $\beta u$ and $\gamma v$ and therefore $\delta w$. If $\beta \neq 0$, then $u$ and $w$ are in $I$. If $\gamma \neq 0$, $v$ and $w$ are in $I$. If $\beta = \gamma = 0$, $I = (w)$. Thus there are precisely five nonzero ideals: $(w)$, $(u, v)$, $(v, w)$, $(u, v, w)$, $(u, v, w, e)$. Since $w$ is not in $M_R$ or $M_L$, they are zero.

We now return to the commutative case but drop DCC. Then $M = \{x: x \text{ is in } xA\}$ and thus $M$ contains all idempotents. In (1) all $M'_i$ = the corresponding $M_i$.

**Theorem 7.** If $A$ is commutative then $M = 0$ if and only if $A$ is isomorphic to a subdirect sum of subdirectly irreducible rings with an absolute divisor of zero in their minimal ideals. That is, they are of type $\beta$ [7].

From [7] we know that a commutative subdirectly irreducible ring with the ascending chain condition is either nilpotent or has a unity element. Thus we have

**Theorem 8.** If $A$ is commutative with ACC then $M = 0$ if and only if $A$ is isomorphic to a subdirect sum of nilpotent subdirectly irreducible rings.

Though Theorems 7 and 8 seem to yield radical-like results, this may be misleading. Let $A$ be the ring of even integers. It has ACC but not DCC. Also $M = 0$ and $A$ is isomorphic to a subdirect sum of nilpotent rings, namely

$$A = (A/(4), A/(8), \ldots, A/(2^n), \ldots)$$

where $A/(2^n)$ has $2^{n-1}$ as an absolute divisor of zero and is nilpotent for every $n$. However $J = 0$ and therefore $A$ is isomorphic to a subdirect sum of fields, namely

$$A = (A/(6), A/(10), A/(14), \ldots, A/(2p), \ldots)$$
where $p$ is a prime. Thus $A$ may be semi-simple and still have $M = 0$. This example also shows that DCC is necessary to obtain nilpotence. To see that ACC is necessary to obtain a subdirect sum of nilpotent rings, we may consider the example in [7] which is commutative, subdirectly irreducible, has neither chain condition, has $M = 0$ and is not nilpotent.

3. **The cases** $A = M_R$, $A = M_L$, $A = M_R = M_L$. In studying the existence of right, left and two sided unities, Baer [1; 2; 3], concerned himself to some extent with right and left $D$-regularity. We summarize some of his results in the language of $M_R$, $M_L$ and $M$:

With DCC on one-sided ideals:

1. $A = M_R$ if and only if $A$ has a right unity.
2. $A = M_R = M_L$ if and only if $A$ has a unity.
3. $A$ commutative, $A = M$ if and only if $A$ has a unity.

If $A - J$ has a unity or if $A - J$ has DCC on one-sided ideals:

2a. $A = M_R = M_L$ if and only if $A$ has a unity.
3a. $A$ commutative, $A = M$ if and only if $A$ has a unity.

However la needed some strengthening:

1b. $A$ has a right unity if and only if $A = M_R$ and when $A = J + Ax$, $x$ must be in $Ax$.

Baer also proved, for a ring with DCC on one-sided ideals:

1. $A$ has a right unity if and only if $A$ has a non-right-zero divisor, i.e. an element $x$ such that $yx = 0$ implies $y = 0$.

Let $A$ be a ring with ACC on left ideals. Then, as is well known, every left ideal of $A$ is finitely generated and in particular $M_R = \{ \sum^n_i n_i a_i + x_i a_i \}$, $x_i$ in $A$, $a_i$ in $M_R$, $n_i$ integers. By the corollary to Lemma 5, there exists an element $e$ in $A$ such that $M_R = M_Re$, pointwise. If we assume $M_R = A$, $e$ is a right unity element.

**Theorem 9.** If $A$ has ACC on left ideals, then $A$ has a right unity if and only if $A = M_R$.

Passing now to rings without chain conditions, we first prove

**Lemma 6.** $A$ has a left unity if and only if there exists an element $x$ in $A$ such that $x$ is in $xA$ and $x$ is not a left zero divisor.

**Proof.** If $A$ has a left unity $f$, then $f$ is in $fA$, and if $fy = 0$ then clearly $y = 0$. Conversely if $x$ is in $xA$, $x = xe$, then for every $y$, $x(y - ey) = 0$ and since $x$ is not a left zero divisor, $y = ey$, $e$ is a left unity.

Note that if $x$ were also not a right zero divisor, then $A$ would have a unity. For $x = ex = xe$ and $(y - ye)x = 0$ yields $y = ye$ for every $y$.

**Corollary.** $A$ has a unity if and only if either $M_R$ or $M_L$ has a non-zero-divisor.
We thus have for rings without chain conditions:

**Theorem 10.**

a. If \( A = M_R \), then \( A \) has a left unity if and only if \( A \) has a non-left-zero-divisor; \( A \) has a unity if and only if \( A \) has a non-zero-divisor.

b. If \( A = M_L \), \( A \) has a right unity if and only if \( A \) has a non-right-zero-divisor.

c. If \( A = M_R = M_L \), then \( A \) has a unity if and only if \( A \) has a non-zero-divisor if and only if \( A \) has a non-right and a non-left-zero-divisor.

Note that then if \( A = M_R = M_L \) and if \( A \) has neither a right nor a left unity, then every element of \( A \) is a two-sided divisor of zero.

In the commutative case, when \( A = M \) it is thus clear that \( A \) has a unity if and only if \( A \) has a non-zero-divisor. We can obtain more. For when \( A \) is expressed as a subdirect sum of subdirectly irreducible rings \( A_i \), each \( A_i \) must be equal to its \( M_i \): Let \( x \) be any element of \( A_i \).

It must appear in the expansion of some element, say \( x = (x_1, \cdots, x_i, \cdots) \). Since \( A = M \), there exists an element \( y \) in \( A \) such that \( xy = x \). Let \( y = (y_1, \cdots, y_i, \cdots) \). Then \( x_i y_i = x_i \) and \( x_i \) is in \( M_i \), \( A_i = M_i \). However from [7], it is clear that if a commutative subdirectly irreducible ring is equal to its maximal \( D \)-regular ideal, it has a unity element. In fact it is either a field or has a unity and is a field modulo its set (an ideal) of zero-divisors. If in addition it has ACC, it is either a field or has a unity and is a field modulo its maximal nilideal. Thus we have:

**Theorem 11.** If \( A \) is a commutative ring and \( A = M \), then \( A \) is isomorphic to a subdirect sum of subdirectly irreducible rings each with a unity. Some are fields and others are fields modulo the ideal of zero-divisors. If \( A \) has ACC, the latter set are fields modulo their maximal nilideals.

Of course \( A \) itself may not have a unity, for let \( A \) be the weak direct sum of an infinite number of fields. Every element of \( A \) is a zero-divisor and \( A \) has no unity.

In summary we have:

\( A = M_R = M_L \) if and only if \( A \) has a unity and any one of the five following conditions: DCC on one-sided ideals; ACC on one-sided ideals; a unity element in \( A - J \); a non-zero-divisor; a non-left and non-right-zero-divisor.

\( A = M_R \) if and only if \( A \) has a right unity and one of the following three conditions: DCC on right ideals; ACC on left ideals; \( A - J \) has a unity element and \( A = J + Ax \) implies that \( x \) must be in \( Ax \).
$A = M_R$ implies that $A$ has a left unity, if it has a non-left-zero-divisor.

$A = M_R$ if and only if $A$ has a unity, if $A$ has a non-zero-divisor.

**Bibliography**


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